



ON A NEW SUBCLASS OF M-FOLD SYMMETRIC BIUNIVALENT FUNCTIONS EQUIPPED WITH SUBORDINATE CONDITIONS

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ABSTRACT. In this paper, we introduce a new subclass of biunivalent function class Σ in which both $f(z)$ and $f^{-1}(z)$ are m -fold symmetric analytic functions. For functions of the subclass introduced in this paper, we obtain the coefficient bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ and also study the Fekete–Szegő functional estimate for this class. Consequences of the results are also discussed.

1. INTRODUCTION

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. Let S be the subclass of A consisting of functions, which are analytic and univalent in \mathbb{U} .

The Keobe one-quarter theorem [8] states that, the range of every function of the class S contains the disk $\{w : |w| < 1/4\}$. Therefore, every $f \in S$ has an inverse function f^{-1} satisfying

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad (|w| < r_0(f); r_0(f) \geq 1/4).$$

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The inverse of $f(z)$ has a series expansion in some disc about the origin of the form

$$f^{-1}(w) = w + A_2w^2 + A_3w^3 + \dots . \tag{1.2}$$

A function $f(z)$, which is univalent in a neighborhood of the origin, and its inverse satisfy the condition $f(f^{-1}(w)) = w$ using (1.2) yields

$$w = f^{-1}(w) + a_2(f^{-1}(w))^2 + a_3(f^{-1}(w))^3 + \dots , \tag{1.3}$$

and now using (1.3), we get the following result:

$$g(w) = f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \dots . \tag{1.4}$$

An analytic function $f(z)$ is said to be biunivalent in \mathbb{U} if both $f(z)$ and $f^{-1}(z)$ are univalent in \mathbb{U} . The class of analytic biunivalent function in \mathbb{U} is denoted by Σ .

For a brief history and interesting examples of functions in the class Σ ; see the pioneering work on this subject by Srivastava et al. [18], which has apparently revived the study of biunivalent functions in recent years. From the work of Srivastava et al. [18], we choose to recall the following examples of functions in the class Σ :

$$\frac{z}{1-z}, \quad -\log(1-z), \quad \frac{1}{2} \log\left(\frac{1+z}{1-z}\right),$$

and so on. However, the familiar Koebe function is not a member of the biunivalent function class Σ . Such other common examples of functions in S as

$$z - \frac{z^2}{2} \quad \text{and} \quad \frac{z}{1-z^2}$$

are also not members of Σ (see [18]).

If the function f and g are analytic in \mathbb{U} , then f is said to be subordinate to g , written as

$$f(z) \prec g(z) \quad (z \in \mathbb{U})$$

if there exists a Schwarz function $w(z)$, analytic in \mathbb{U} , with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g(w(z)) \quad (z \in \mathbb{U}).$$

Lewin [11] studied the class of biunivalent functions, obtaining the bound 1.51 for the modulus of the second coefficient $|a_2|$. Subsequently, Brannan and Clunie [6] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Later on, Netanyahu [14] showed that $\max |a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Brannan and Taha [7] introduced certain subclasses of the biunivalent function class Σ similar to the familiar subclasses $S^*(\beta)$ and $K(\beta)$ of starlike and convex functions of order β ($0 \leq \beta < 1$) in \mathbb{U} , respectively (see [14]). The classes $S_\Sigma^*(\beta)$ and $K_\Sigma(\beta)$ of bistarlike functions of order β in \mathbb{U} and biconvex functions of order β in \mathbb{U} , corresponding to the function classes $S^*(\beta)$ and $K(\beta)$, were also introduced analogously. For each of the function classes $S_\Sigma^*(\beta)$ and $K_\Sigma(\beta)$, they found nonsharp estimates for the initial coefficients.

Recently, motivated substantially by the aforementioned pioneering work on this subject by Srivastava et al. [18], many authors investigated the coefficient bounds for various subclasses of biunivalent functions (see, for example, [1], [2], [3], [4], [10], [13], [15], [20], [22], and [23]). Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. The coefficient estimate problem for each of the coefficients

$$|a_n| \quad (n \in \mathbb{N} \setminus \{1, 2\}, \mathbb{N} = \{1, 2, 3, \dots\})$$

is still an open problem.

For each function f in S , the function h given by

$$h(z) = \sqrt[m]{f(z^m)} \quad (m \in \mathbb{N})$$

is univalent and maps the unit disk \mathbb{U} into a region with m -fold symmetry. A function is said to be m -fold symmetric (see [16]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \quad (m \in \mathbb{N}, z \in \mathbb{U}). \tag{1.5}$$

We denote the class of m -fold symmetric univalent functions by S_m , which are normalized by the above series expansion (1.5). In fact the functions in the class S are one fold symmetric (that is $m = 1$). Analogous to the concept of m -fold symmetric univalent functions, one can think of the concept of m -fold symmetric biunivalent function in a natural way. Each function f in the class Σ generates an m -fold symmetric biunivalent function for each positive integer m . The normalized form of f is given as (1.5) and f^{-1} is given by as follows:

$$g(w) = w - a_{m+1} z^{m+1} + [(m + 1)a_{m+1}^2 - a_{2m+1}] w^{2m+1} - \left[\frac{1}{2}(m + 1)(3m + 2)a_{m+1}^3 - (3m + 2)a_{m+1}a_{2m+1} + a_{3m+1} \right] w^{3m+1} + \dots, \tag{1.6}$$

where $f^{-1} = g$. We denote the class of m -fold symmetric biunivalent functions by Σ_m . For $m = 1$, the formula (1.6) coincides with the function (1.4) of the class Σ . Some examples of m -fold symmetric biunivalent functions are given here below:

$$\left(\frac{z^m}{1 - z^m} \right)^{\frac{1}{m}}, \quad [-\log(1 - z^m)]^{\frac{1}{m}}, \quad \left[\frac{1}{2} \log \left(\frac{1 + z^m}{1 - z^m} \right)^{\frac{1}{m}} \right].$$

Here in this paper, we also denote P the class of analytic functions of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \dots,$$

such that

$$R(p(z)) > 0 \quad (z \in \mathbb{U}).$$

In view of the work of Pommerenke [16] the m -fold symmetric function p in the class P is of the form

$$p(z) = 1 + c_m z^m + c_{2m} z^{2m} + c_{3m} z^{3m} + \dots. \tag{1.7}$$

Let ϕ be an analytic function with positive real part in \mathbb{U} , with $\phi(0) = 1$ and $\phi'(0) > 0$. Also, let $\phi(\mathbb{U})$ be starlike with respect to one and symmetric with respect to the axis. Thus, ϕ has the Taylor series expansion

$$\phi(z) = 1 + B_1z + B_2z^2 + B_3z^3 + \dots \quad (B_1 > 0). \tag{1.8}$$

Suppose that $u(z)$ and $v(w)$ are analytic in the unit disk \mathbb{U} with $u(0) = v(0) = 0$, $|u(z)| < 1$, and $|v(w)| < 1$.

We suppose that

$$u(z) = b_mz^m + b_{2m}z^{2m} + b_{3m}z^{3m} + \dots \quad (|z| < 1) \tag{1.9}$$

and

$$v(w) = c_mz^m + c_{2m}z^{2m} + c_{3m}z^{3m} + \dots \quad (|w| < 1). \tag{1.10}$$

It is well known that

$$|b_m| \leq 1, |b_{2m}| \leq 1 - |b_m|^2, |c_m| \leq 1, |c_{2m}| \leq 1 - |c_m|^2. \tag{1.11}$$

By simple computations

$$\phi(u(z)) = 1 + B_1b_mz^m + (B_1b_{2m} + B_2b_m^2)z^{2m} + \dots \quad (|z| < 1) \tag{1.12}$$

and

$$\phi(v(w)) = 1 + B_1c_mz^m + (B_1c_{2m} + B_2c_m^2)z^{2m} + \dots \quad (|w| < 1). \tag{1.13}$$

Babalola [5] defined the class $L_\lambda(\beta)$ of λ -pseudo-starlike functions of order β as below.

Definition 1.1. Let $f \in A$; suppose that $0 \leq \beta < 1$ and that $\lambda \geq 1$ is real. Then $f(z) \in L_\lambda(\beta)$ of λ -pseudo-starlike functions of order β in the unit disk if and only if

$$Re \frac{z [f'(z)]^\lambda}{f(z)} > \beta.$$

Babalola [5] proved that, all pseudo-starlike functions are Bazilevic of type $(1 - \frac{1}{\lambda})$ order $\frac{1}{\beta\lambda}$ and univalent in open unit disk \mathbb{U} .

We now introduce the following subclass of m -fold symmetric biunivalent function class Σ_m .

Definition 1.2. A function $f \in \Sigma_m$ said to be in the class $S_{\Sigma,m}^\lambda(\phi)$, if the following subordination conditions hold:

$$\frac{z [f'(z)]^\lambda}{f(z)} \prec \phi(z) \tag{1.14}$$

and

$$\frac{w [g'(w)]^\lambda}{g(w)} \prec \phi(w),$$

where $g = f^{-1}$ and $\lambda \geq 1$.

For various special choices of the function ϕ and for the case when $m = 1$, our function class $S_{\Sigma,m}^\lambda(\phi)$ reduces to the following known classes:

- (1) Taking $m = 1$, the function class is given by

$$S_{\Sigma,m}^\lambda(\phi) \equiv S_{\Sigma,1}^\lambda(\phi) \equiv S_\Sigma^\lambda(\phi).$$

- (2) For $m = 1$ and $\phi(z) = \left(\frac{1+z}{1-z}\right)^\alpha$ ($0 < \alpha \leq 1$), the function class given by

$$S_{\Sigma,m}^\lambda(\phi) \equiv S_{\Sigma,1}^\lambda \left(\left(\frac{1+z}{1-z} \right)^\alpha \right)$$

was studied by Joshi et al. [10].

- (3) For $m = 1$ and $\phi(z) = \left(\frac{1+(1-2\beta)z}{1-z}\right)$ ($0 \leq \beta < 1$), the function class given by

$$S_{\Sigma,m}^\lambda(\phi) \equiv S_{\Sigma,1}^\lambda \left(\frac{1+(1-2\beta)z}{1-z} \right)$$

was studied by Joshi et al. [10].

Motivated by the work of Ma and Minda [12] and Srivastava et al. [19], we introduce a new subclass of m -fold symmetric biunivalent functions. We obtain the coefficients bounds for $|a_{m+1}|$ and $|a_{2m+1}|$ and also the Fekete–Szegő functional estimate for the subclass. The results improve the earlier results of Joshi et al. [10].

2. COEFFICIENT ESTIMATES

We begin this section by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $S_{\Sigma,m}^\lambda(\phi)$ proposed by Definition 1.2.

Theorem 2.1. *Let the function f given by (1.5) be in the class $S_{\Sigma,m}^\lambda(\phi)$. Then*

$$|a_{m+1}| \leq \frac{B_1 \sqrt{2B_1}}{\sqrt{2(\lambda m + \lambda - 1)^2 B_1 + |(m^2 \lambda^2 + \lambda m^2 + 2m \lambda^2 - \lambda m + \lambda^2 - 2\lambda - m + 1) B_1^2 - 2(\lambda m + \lambda - 1)^2 B_2|}} \tag{2.1}$$

$$|a_{2m+1}| \leq$$

$$\begin{cases} \frac{B_1}{|2m\lambda + \lambda - 1|}, & B_1 < \frac{2(\lambda m + \lambda - 1)^2}{(m+1)|2m\lambda + \lambda - 1|}, \\ \left((m+1) - \frac{2(m\lambda + \lambda - 1)^2}{|2m\lambda + \lambda - 1| B_1} \right) \frac{B_1^3}{2(m\lambda + \lambda - 1)^2 B_1 + |(m^2 \lambda^2 + m^2 \lambda + 2m \lambda^2 - m \lambda + \lambda^2 - 2\lambda - m + 1) B_1^2 - 2(m\lambda + \lambda - 1)^2 B_2|} \\ + \frac{B_1}{|2\lambda m + \lambda - 1|}, & B_1 \geq \frac{2(\lambda m + \lambda - 1)^2}{(m+1)|2m\lambda + \lambda - 1|}. \end{cases} \tag{2.2}$$

Proof. Let $f \in S_{\Sigma,m}^\lambda$ and $g = f^{-1}$. Then there are analysis functions $u : \mathbb{U} \rightarrow \mathbb{U}$ and $v : \mathbb{U} \rightarrow \mathbb{U}$, with

$$u(0) = v(0) = 0,$$

satisfying the following conditions:

$$\frac{z[f'(z)]^\lambda}{f(z)} = \phi(u(z)) \tag{2.3}$$

and

$$\frac{w[g'(w)]^\lambda}{g(w)} = \phi(v(w)). \tag{2.4}$$

Comparing the corresponding coefficients of (2.3) and (2.4) yields

$$(m\lambda + \lambda - 1)a_{m+1} = B_1 b_m, \tag{2.5}$$

$$\left[\lambda(m+1) \left(\frac{(\lambda-1)(m+1)}{2} - 1 \right) + 1 \right] a_{m+1}^2 + (2m\lambda + \lambda - 1)a_{2m+1} = B_1 b_{2m} + B_2 b_m^2, \tag{2.6}$$

$$-(m\lambda + \lambda - 1)a_{m+1} = B_1 c_m, \tag{2.7}$$

and

$$\left[\lambda(m+1) \left(\frac{(\lambda-1)(m+1)}{2} + 2m \right) - m \right] a_{m+1}^2 - (2m\lambda + \lambda - 1)a_{2m+1} = B_1 c_{2m} + B_2 c_m^2. \tag{2.8}$$

It implies from (2.5) and (2.7) that

$$c_m = -b_m. \tag{2.9}$$

By adding (2.6) and (2.8), further computation using (2.5) and (2.9) lead to

$$[(m^2\lambda^2 + \lambda m^2 + 2m\lambda^2 - \lambda m + \lambda^2 - 2\lambda - m + 1)B_1^2 - 2(\lambda m + \lambda - 1)^2 B_2] a_{m+1}^2 = B_1^3 (b_{2m} + c_{2m}). \tag{2.10}$$

Using (2.9) and (2.10), together with (1.11), yield

$$|(m^2\lambda^2 + \lambda m^2 + 2m\lambda^2 - \lambda m + \lambda^2 - 2\lambda - m + 1)B_1^2 - 2(\lambda m + \lambda - 1)^2 B_2| |a_{m+1}|^2 \leq 2B_1^3 (1 - |b_m|^2). \tag{2.11}$$

Equations (2.5) and (2.11) give the desired estimate on $|a_{m+1}|$ as asserted in (2.1). By subtracting (2.8) from (2.6), we obtain

$$2(2m\lambda + \lambda - 1)a_{2m+1} = (m+1)(2m\lambda + \lambda - 1)a_{m+1}^2 + B_1(b_{2m} - c_{2m}). \tag{2.12}$$

From (1.11), (2.5), (2.8), and (2.12), it follows that

$$\begin{aligned} |a_{2m+1}| &\leq \frac{(m+1)}{2} |a_{m+1}|^2 + \frac{B_1}{2|2m\lambda + \lambda - 1|} (|b_{2m}| + |c_{2m}|) \\ &\leq \frac{(m+1)}{2} |a_{m+1}|^2 + \frac{B_1}{|2m\lambda + \lambda - 1|} (1 - |b_m|^2) \\ &= \left(\frac{m+1}{2} - \frac{(m\lambda + \lambda - 1)^2}{|2m\lambda + \lambda - 1| B_1} \right) |a_{m+1}|^2 + \frac{B_1}{|2m\lambda + \lambda - 1|}, \end{aligned}$$

which implies the assertion (2.2). □

For the case of one-fold symmetric function, Theorem 2.1 reduces to Corollary 2.2 below.

Corollary 2.2. *Let the function f given by (1.5) be in the class $S_{\Sigma}^{\lambda}(\phi)$. Then*

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{(2\lambda - 1) [(2\lambda - 1)B_1 + |\lambda B_1^2 - (2\lambda - 1)B_2]]}} \tag{2.13}$$

and

$$|a_3| \leq \begin{cases} \frac{B_1}{3\lambda - 1}, & B_1 < \frac{(2\lambda - 1)^2}{3\lambda - 1}, \\ \left(1 - \frac{(2\lambda - 1)^2}{(3\lambda - 1)B_1}\right) \frac{B_1^3}{(2\lambda - 1)^2 B_1 + (2\lambda^2 - \lambda)B_1^2 - (2\lambda - 1)^2 B_2} + \frac{B_1}{3\lambda - 1}, & B_1 \geq \frac{(2\lambda - 1)^2}{3\lambda - 1}. \end{cases} \tag{2.14}$$

Remark 2.3. For $f \in S_{\Sigma}^{\lambda}(\phi)$, the function ϕ is given by

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \dots \quad (0 < \alpha \leq 1),$$

and so $B_1 = 2\alpha$ and $B_2 = 2\alpha^2$. Hence Corollary 2.2 reduces to an improved results of Joshi et al. [10].

On the other hand when

$$\phi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} = 1 + 2(1 - \beta)z + 2(1 - \beta)^2 z^2 + \dots \quad (0 \leq \beta < 1),$$

$B_1 = B_2 = 2(1 - \beta)$, and thus Corollary 2.2 reduces to the improved results of Joshi et al. [10].

For the case of one-fold symmetric functions with $\lambda = 1$, the class reduces to the strongly starlike functions; the function ϕ is given by

$$\phi(z) = \left(\frac{1+z}{1-z}\right)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + \dots \quad (0 < \alpha \leq 1), \tag{2.15}$$

which gives

$$B_1 = 2\alpha \text{ and } B_2 = 2\alpha^2.$$

Hence, Theorem 2.1 gives the following corollary.

Corollary 2.4. *Let the function f given by (1.5) be in the class $S_{\Sigma,1}^1\left(\left(\frac{1+z}{1-z}\right)^{\alpha}\right)$. Then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{1 + \alpha}} \tag{2.16}$$

and

$$|a_3| \leq \begin{cases} \alpha, & 0 < \alpha \leq \frac{1}{4}, \\ \frac{5\alpha^2}{1 + \alpha}, & \frac{1}{4} < \alpha \leq 1. \end{cases} \tag{2.17}$$

For the case of one-fold symmetric functions with $\lambda = 1$, the class reduces to the strongly starlike functions, and the function ϕ is given by

$$\phi(z) = 1 + 2(1 - \beta)z + 2(1 - \beta)^2 z^2 + \dots \quad (0 \leq \beta < 1);$$

so that

$$B_1 = B_2 = 2(1 - \beta).$$

Corollary 2.5. *Let the function f given by (1.5) be in the class $S_{\Sigma,1}^1 \left(\frac{1+(1-2\beta)z}{1-z} \right)$. Then*

$$|a_2| \leq \frac{2(1-\beta)}{\sqrt{1+|1-2\beta|}} \tag{2.18}$$

and

$$|a_3| \leq \begin{cases} \frac{5-6\beta}{2}, & 0 \leq \beta < \frac{3}{4}, \\ 1-\beta, & \frac{3}{4} \leq \beta < 1. \end{cases} \tag{2.19}$$

3. FEKETE–SZEGÖ PROBLEM

The classical Fekete–Szegő inequality, presented by means of Loewner’s method, for the coefficients of $f \in S$, is

$$|a_3 - \mu a_2^2| \leq 1 + 2 \exp(-2\mu/(1-\mu)) \text{ for } \mu \in [0, 1).$$

As $\mu \rightarrow 1^-$, we have the elementary inequality $|a_3 - a_2^2| \leq 1$. Moreover, the coefficient functional

$$\Phi_\mu(f) = a_3 - \mu a_2^2$$

on the normalized analytic functions f in the unit disk \mathbb{U} plays an important role in function theory. The problem of maximizing the absolute value of the functional $\Phi_\mu(f)$ is called the Fekete–Szegő problem, see [9].

In this section, we aim to provide Fekete–Szegő inequalities for functions in the class $S_{\Sigma,m}^\lambda(\phi)$. These inequalities are given in the following theorem.

Theorem 3.1. *Let the function $f(z)$, given by (1.5), be in the class $S_{\Sigma,m}^\lambda(\phi)$. Then*

$$|a_{2m+1} - \mu a_{m+1}^2| \leq \begin{cases} \frac{B_1}{|2m\lambda + \lambda - 1|}, & 0 \leq |h(\mu)| < \frac{1}{2|2m\lambda + \lambda - 1|} \\ 2B_1 |h(\mu)|, & |h(\mu)| \geq \frac{1}{2|2m\lambda + \lambda - 1|}, \end{cases} \tag{3.1}$$

where

$$h(\mu) = \frac{B_1^2(m+1-2\mu)}{2[(m^2\lambda^2 + m^2\lambda + 2m\lambda^2 - m\lambda + \lambda^2 - 2\lambda - m + 1)B_1^2 - 2(m\lambda + \lambda - 1)^2B_2]}.$$

Proof. From the equation (2.10), we get

$$a_{m+1}^2 = \frac{B_1^3(b_{2m} + c_{2m})}{(m^2\lambda^2 + m^2\lambda + 2m\lambda^2 - m\lambda + \lambda^2 - 2\lambda - m + 1)B_1^2 - 2(m\lambda + \lambda - 1)^2B_2}. \tag{3.2}$$

By subtracting (2.6) from (2.8), we get

$$a_{2m+1} = \frac{(m+1)}{2} a_{m+1}^2 + \frac{B_1(b_{2m} - c_{2m})}{2(2m\lambda + \lambda - 1)}. \tag{3.3}$$

From equations (3.2) and (3.3), we obtain

$$a_{2m+1} - \mu a_{m+1}^2 = B_1 \left[\left(h(\mu) + \frac{1}{2(2m\lambda + \lambda - 1)} \right) b_{2m} + \left(h(\mu) - \frac{1}{2(2m\lambda + \lambda - 1)} \right) c_{2m} \right],$$

where

$$h(\mu) = \frac{B_1^2(m+1-2\mu)}{2[(m^2\lambda^2+m^2\lambda+2m\lambda^2-m\lambda+\lambda^2-2\lambda-m+1)B_1^2-2(m\lambda+\lambda-1)^2B_2]}.$$

All B_i are real and $B_1 > 0$, which implies the assertion equation (3.1). \square

For the case of one-fold symmetric functions, Theorem 3.1 reduces to the following Corollary 3.2.

Corollary 3.2. *Let the function f given by (1.5) be in the class $S_{\Sigma,1}^\lambda(\phi)$. Then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{B_1}{3\lambda-1}, & 0 \leq |h(\mu)| < \frac{1}{2(3\lambda-1)}, \\ 2B_1|h(\mu)|, & |h(\mu)| \geq \frac{1}{2(3\lambda-1)}, \end{cases}$$

where

$$h(\mu) = \frac{B_1^2(1-\mu)}{2(2\lambda-1)[\lambda B_1^2 - (2\lambda-1)B_2]}.$$

Taking $\mu = 1$ and $\mu = 0$ in Theorem 3.1, we have the following corollaries.

Corollary 3.3. *Let the function f given by (1.5) be in the class $S_{\Sigma,m}^\lambda(\phi)$. Then*

$$|a_{2m+1} - a_{m+1}^2| \leq \begin{cases} \frac{B_1}{|2m\lambda + \lambda - 1|}, & \frac{B_2}{B_1^2} \in (-\infty, \rho_1) \cup (\rho_2, \infty), \\ \frac{B_1^3(m-1)}{|(m^2\lambda^2+m^2\lambda+2m\lambda^2-m\lambda+\lambda^2-2\lambda-m+1)B_1^2-2(m\lambda+\lambda-1)^2B_2|}, & \\ \frac{B_2}{B_1^2} \in \left(\rho_1, \frac{m^2\lambda^2+m^2\lambda+2m\lambda^2-m\lambda+\lambda^2-2\lambda-m+1}{2(m\lambda+\lambda-1)^2}\right) \cup \left(\frac{m^2\lambda^2+m^2\lambda+2m\lambda^2-m\lambda+\lambda^2-2\lambda-m+1}{2(m\lambda+\lambda-1)^2}, \rho_2\right), \end{cases}$$

where

$$\rho_1 = \frac{m^2\lambda^2 - m^2\lambda + 2m\lambda^2 + \lambda^2 - \lambda}{2(m\lambda + \lambda - 1)^2}$$

and

$$\rho_2 = \frac{m^2\lambda^2 + 3m^2\lambda + 2m\lambda^2 - 2m\lambda + \lambda^2 - 3\lambda - 2m + 2}{2(m\lambda + \lambda - 1)^2}.$$

For the case of one-fold symmetric functions, Corollary 3.3 reduces to the following corollary.

Corollary 3.4. *Let the function f given by (1.5) be in the class $S_{\Sigma,1}^\lambda(\phi)$. Then*

$$|a_3 - a_2^2| \leq \frac{B_1}{3\lambda - 1}.$$

Also, letting $\lambda = 1$, we obtain

$$|a_3 - a_2^2| \leq \frac{B_1}{2}.$$

Corollary 3.5. *Let the function f given by (1.5) be in the class $S_{\Sigma,m}^\lambda(\phi)$. Then*

$$|a_{2m+1}| \leq \begin{cases} \frac{B_1}{|2m\lambda + \lambda - 1|}, & \frac{B_2}{B_1^2} \in (-\infty, \sigma_1) \cup (\sigma_2, \infty), \\ \frac{B_1^3(m+1)}{|(m^2\lambda^2+m^2\lambda+2m\lambda^2-m\lambda+\lambda^2-2\lambda-m+1)B_1^2-2(m\lambda+\lambda-1)^2B_2|}, \\ \frac{B_2}{B_1^2} \in \left(\sigma_1, \frac{m^2\lambda^2+m^2\lambda+2m\lambda^2-m\lambda+\lambda^2-2\lambda-m+1}{2(m\lambda+\lambda-1)^2}\right) \cup \left(\frac{m^2\lambda^2+m^2\lambda+2m\lambda^2-m\lambda+\lambda^2-2\lambda-m+1}{2(m\lambda+\lambda-1)^2}, \sigma_2\right), \end{cases}$$

where

$$\sigma_1 = \frac{m^2\lambda^2 - m^2\lambda + 2m\lambda^2 - 4m\lambda + \lambda^2 - 3\lambda + 2}{2(m\lambda + \lambda - 1)^2}$$

and

$$\sigma_2 = \frac{m^2\lambda^2 + 3m^2\lambda + 2m\lambda^2 + 2m\lambda + \lambda^2 + \lambda - 2m}{2(m\lambda + \lambda - 1)^2}.$$

Corollary 3.6. *Let the function f given by (1.5) be in the class $S_{\Sigma,1}^\lambda(\phi)$. Then*

$$|a_3| \leq \begin{cases} \frac{B_1}{3\lambda-1}, & \frac{B_2}{B_1^2} \in \left(-\infty, \frac{2\lambda^2-4\lambda+1}{(2\lambda-1)^2}\right) \cup \left(\frac{2\lambda^2+2\lambda-1}{(2\lambda-1)^2}, \infty\right), \\ \frac{B_1^3}{(2\lambda-1)[\lambda B_1^2-(2\lambda-1)B_2]}, & \frac{B_2}{B_1^2} \in \left(\frac{2\lambda^2-4\lambda+1}{(2\lambda-1)^2}, \frac{\lambda}{2\lambda-1}\right) \cup \left(\frac{\lambda}{2\lambda-1}, \frac{2\lambda^2+2\lambda-1}{(2\lambda-1)^2}\right). \end{cases}$$

For the cases of one-fold symmetric functions and $\lambda = 1$, Corollary 3.6 reduces to the following corollary.

Corollary 3.7 (see [21]). *Let the function f given by (1.5) be in the class $S_{\Sigma,1}^1(\phi)$. Then*

$$|a_3| \leq \begin{cases} \frac{B_1}{2}, & \frac{B_2}{B_1^2} \in (-\infty, -1) \cup (3, \infty), \\ \frac{B_1^3}{B_1^2-B_2}, & \frac{B_2}{B_1^2} \in (-1, 1) \cup (1, 3). \end{cases}$$

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