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EXPANDING THE APPLICABILITY OF GENERALIZED HIGH CONVERGENCE ORDER METHODS FOR SOLVING EQUATIONS

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Abstract. The local convergence analysis of iterative methods is important since it indicates the degree of difficulty for choosing initial points. In the present study we introduce generalized three step high order methods for solving nonlinear equations. The local convergence analysis is given using hypotheses only on the first derivative, which actually appears in the methods in contrast to earlier works using hypotheses on higher derivatives. This way we extend the applicability of these methods. The analysis includes computable radius of convergence as well as error bounds based on Lipschitz-type conditions, which is not given in earlier studies. Numerical examples conclude this study.

1. Introduction

The task of solving equations, derived by concrete problems through mathematical modeling, has recently gained a greater importance due to exponential advancement of computer hardware and software. Let $F : D \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a Fréchet-differentiable operator, where $\mathcal{B}_1$ and $\mathcal{B}_2$ are Banach spaces and D is a nonempty open convex subset of $\mathcal{B}_1$. Consider the problem of finding a locally unique solution $x^* \in D$ of equation

$$F(x) = 0. \tag{1.1}$$

Most solution methods of equation (1.1) are iterative, since a solution x^* is given in closed form only in special cases. Motivated by the single step, quadratically

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convergent Newton's method is defined, for each $n = 0, 1, 2, \ldots$, by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n), \tag{1.2}$$

where x_0 is an initial point. Numerous authors have introduced multi-step methods in order to increase the convergence order. In particular three step methods have been introduced in the special case when $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^i$ (i a natural number) to solve nonlinear systems [2, 4, 12–16, 19–27]. We introduce in a Banach space setting generalized three step methods defined, for each $n = 0, 1, 2, \ldots$, by

$$\begin{aligned} y_n &= x_n - \alpha F'(x_n)^{-1}F(x_n)' \\ z_n &= \varphi(x_n, y_n), \\ x_{n+1} &= \psi(x_n, y_n, z_n), \end{aligned} \tag{1.3}$$

where x_0 is an initial point, $\alpha \in S$, where $S = \mathbb{R}$ or $S = \mathbb{C}$, and $\varphi : D^2 \longrightarrow \mathcal{B}_1, \psi : D^3 \longrightarrow \mathcal{B}_1$ are iteration operators. Usually φ is an iteration operator of convergence order $p \geq 2$. Numerous popular iterative methods are special cases of method (1.3) [1–27] (see Section 3).

The local convergence analysis usually involves Taylor expansions and conditions on higher order derivatives not appearing in these methods. Moreover, these approaches do not provide computable radius of convergence and error estimates on the distances $\|x_n - x^*\|$. Therefore the initial point is a shot in the dark. These problems limit the usage of these methods. That is why in the present study using only conditions on the first derivative, we address the preceding problems in the more general setting of methods (1.3) and Banach space.

We find computable radii of convergence as well as error bounds on the distances based on Lipschitz- type conditions. The order of convergence is found using computable order of convergence (COC) or approximate computational order of convergence (ACOC) [24] (see Remark 2.2) that do not require usage of higher order derivatives. This way we expand the applicability of three step method (1.3) under weak conditions.

The rest of the study is organized as follows: Section 2 contains the local convergence of method (1.3), where in the concluding Section 3 applications and numerical examples can be found.

2. Local convergence analysis

The local convergence analysis is based on some parameters and scalar functions. Let $w_0 : [0, \infty) \longrightarrow [0, \infty)$ be a continuous and nondecreasing function satisfying $w_0(0) = 0$ and $\alpha \in S$. Define parameter ρ by

$$\rho = \sup\{t \geq 0 : w_0(t) < 1\}. \tag{2.1}$$

Let also $w : [0, \rho) \longrightarrow [0, \infty), v : [0, \rho) \longrightarrow [0, \infty), g_2 : [0, \rho) \longrightarrow [0, \infty)$, and $\gamma : [0, \rho) \longrightarrow [0, \infty)$ be nondecreasing continuous functions with $w(0) = 0$. Define functions g_1 and h_1 on the interval $[0, \rho)$ by

$$g_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta + |1-\alpha| \int_0^1 v(\theta t)d\theta}{1 - w_0(t)}$$

and

$$h_1(t) = g_1(t) - 1.$$

Suppose that

$$|1-\alpha|v(0) < 1. \tag{2.2}$$

We have, by the definitions of the scalar functions, ρ, and (2.2), that $h_1(0) = |1-\alpha|v(0) - 1 < 0$ and $h_1(t) \longrightarrow +\infty$ as $t \longrightarrow \rho^-$. By applying the intermediate value theorem on function h_1, we deduce that the equation $h_1(t) = 0$ has solutions in $(0, \rho)$. Denote by ρ_1 the smallest such solution. Let $g_2 : [0, \rho)^2 \longrightarrow [0, \infty)$ be a continuous and nondecreasing function. Define function h_2 on the interval $[0, \rho)$ by

$$h_2(t) = g_2(t,t) - 1,\ g_2(t) = g_2(t,t).$$

Suppose that

$$g_2(0, 0) < 1 \tag{2.3}$$

and $h_2(t) \longrightarrow$ a positive number or $+\infty$ as

$$t \longrightarrow \bar{\rho} \le \rho^-. \tag{2.4}$$

We get, by (2.3) and (2.4), that the equation $h_2(t) = 0$ has solutions on $(0, \bar{\rho})$. Denote by ρ_2 the smallest such solution. Let $g_3 : [0, \rho)^3 \longrightarrow [0, \infty)$ be a continuous and nondecreasing function. Define function h_3 on the interval $[0, \rho)$ by

$$h_3(t) = g_3(t,t,t) - 1,\ g_3(t) = g_3(t,t,t).$$

Suppose that

$$g_3(0, 0, 0) < 1 \tag{2.5}$$

and that $h_3(t)$ converges to a positive number or $+\infty$ as

$$t \longrightarrow \bar{\bar{\rho}} \le \rho^-. \tag{2.6}$$

Denote by ρ_3 the smallest solution of the equation $h_3(t) = 0$ in $(0, \bar{\bar{\rho}})$. Define the radius of convergence ρ^* by

$$\rho^* = \min\{\rho_i\}, \qquad i = 1, 2, 3. \tag{2.7}$$

Then, for each $t \in [0, \rho^*)$,

$$0 \le g_i(t) < 1, \qquad i = 1, 2, 3. \tag{2.8}$$

Let $U(u, \varepsilon) = \{x \in \mathcal{B}_1 : \|x - u\| < \varepsilon\}$ for $u \in \mathcal{B}_1$ and $\varepsilon > 0$. Let also $\bar{U}(u, \varepsilon)$ stand for its closure.

Next, we present the local convergence analysis of method (1.3) using the preceding notation.

Theorem 2.1. *Let $F : D \subseteq \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a continuously Fréchet-differentiable operator. Suppose that there exists $x^* \in D$ such that*

$$F(x^*) = 0 \ \text{ and } \ F'(x^*)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1); \tag{2.9}$$

and there exists continuous and nondecreasing function $w_0 : [0, \infty) \longrightarrow [0, \infty)$ with $w_0(0) = 0$ such that

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le w_0(\|x - x^*\|) \qquad \text{for each } x \in D. \tag{2.10}$$

Set $D_0 = D \cap U(x^*, \rho^*)$, where ρ^* is defined by (2.7). There exist continuous operators $\varphi : D_0^2 \longrightarrow \mathcal{B}_1$ and $\psi : D_0^3 \longrightarrow \mathcal{B}_1$ and continuous and nondecreasing functions $w : [0, \rho^*) \longrightarrow [0, \infty), v : [0, \rho^*) \longrightarrow [0, \infty), g_2 : [0, \rho^*)^2 \longrightarrow [0, \infty)$, and $g_3 : [0, \rho^*)^3 \longrightarrow [0, \infty)$ such that, for each $x, y, t \in D_0$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq w(\|x - y\|), \tag{2.11}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq v(\|x - x^*\|), \tag{2.12}$$

$$\|\varphi(x, y) - x^*\| \leq g_2(\|x - x^*\|, \|y - x^*\|)\|x - x^*\|, \tag{2.13}$$

$$\|\psi(x, y, z) - x^*\| \leq g_3(\|x - x^*\|, \|y - x^*\|, \|z - x^*\|)\|x - x^*\|, \tag{2.14}$$

and conditions (2.2)–(2.6) and

$$\bar{U}(x^*, \rho^*) \subseteq D \tag{2.15}$$

hold. Then, sequence $\{x_n\}$, generated by method (1.3), for $x_0 \in U(x^*, \rho^*) - \{x^*\}$, is well defined in $U(x^*, \rho^*)$, and it remains in $U(x^*, \rho^*)$ and converges to x^*; so that, for each $n = 0, 1, 2, \dots$,

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\| < \rho^*, \tag{2.16}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|, \|y_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|, \tag{2.17}$$

and

$$\|x_{n+1} - x^*\| \leq g_3(\|x_n - x^*\|, \|y_n - x^*\|, \|z_n - x^*\|)\|x_n - x^*\| \leq \|x_n - x^*\|. \tag{2.18}$$

Moreover, if, for some $r \geq \rho^*$,

$$\int_0^1 w_0(\theta r)d\theta < 1, \tag{2.19}$$

then the limit point x^* is the only solution of the equation $F(x) = 0$ in $D_1 = D \cap \bar{U}(x^*, r)$.

Proof. We shall show, using the induction, that the sequence $\{x_k\}$ is well defined, remains in $U(x^*, \rho^*)$, and converges to x^*; so that estimate (2.16)–(2.18) hold. By conditions (2.9) and (2.16) and $x \in U(x^*, \rho^*)$, we have in turn that

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq w_0(\|x - x^*\|) \leq w_0(\rho^*) < 1. \tag{2.20}$$

It follows, from (2.20) and the Banach lemma on invertible operators [2, 3, 11], that $F'(x)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$ and

$$\|F'(x)^{-1}F'(x^*)\| \leq \frac{1}{1 - w_0(\|x - x^*\|)}. \tag{2.21}$$

In particular, y_0 exists by the first substep of method (1.3) and (2.21) for $x = x_0$ (since $x_0 \in U(x^*, \rho^*)$). Using the first substep of method (1.3), (2.7), (2.8) (for $i = 2$), (2.9), (2.11), (2.12), and (2.21), we obtain in turn that

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) + (1-\alpha)F'(x_0)^{-1}F(x_0) \\ &= \int_0^1 F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}[F'(x^* + \theta(x_0 - x^*)) - F'(x_0)](x_0 - x^*)d\theta \\ &\quad +(1-\alpha)F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}F(x_0); \end{aligned} \tag{2.22}$$

so

$$\begin{aligned}\|y_0 - x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\| \\ &\quad \times \|F'(x^*)^{-1}\int_0^1 [F'(x^* + \theta(x_0 - x^*)) - F'(x_0)](x_0 - x^*)d\theta\| \\ &\quad +|1-\alpha|\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F(x_0)\| \\ &\leq \frac{\int_0^1 w((1-\theta)\|x_0 - x^*\|)d\theta\|x_0 - x^*\| + \int_0^1 v(\theta\|x_0 - x^*\|)d\theta}{1 - w_0(\|x_0 - x^*\|)}\|x_0 - x^*\| \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| \leq \|x_0 - x^*\| < \rho^*; \end{aligned} \tag{2.23}$$

that is, (2.16) holds for $n = 0$ and $y_0 \in U(x^*, \rho^*)$, where we also use the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F(x_0)\| &= \|F'(x^*)^{-1}(F(x_0) - F(x^*)) \\ &= \|\int_0^1 F'(x^*)^{-1}F'(x^* + \theta(x_0 - x^*))d\theta(x_0 - x^*)\| \\ &\leq \int_0^1 v(\theta\|x_0 - x^*\|)d\theta\|x_0 - x^*\|, \end{aligned} \tag{2.24}$$

since $\|x^* + \theta(x_0 - x^*) - x^*\| = \theta\|x_0 - x^*\| \leq \rho^*$ (i.e., $x^* + \theta(x_0 - x^*) \in U(x^*, \rho^*)$ for each $\theta \in [0, 1]$). By the second substep of method (1.3) for $n = 0$, (2.7), (2.8) (for $i = 2$), (2.13), and (2.23), we get in turn

$$\begin{aligned}\|z_0 - x^*\| &= \|\varphi(x_0, y_0) - x^*\| \\ &\leq g_2(\|x_0 - x^*\|, \|y_0 - x^*\|)\|x_0 - x^*\| \\ &\leq g_2(\|x_0 - x^*\|, \|x_0 - x^*\|)\|x_0 - x^*\| \\ &\leq \|x_0 - x^*\| < \rho^*; \end{aligned} \tag{2.25}$$

so (2.17) holds for $n = 0$ and $z_0 \in U(x^*, \rho^*)$. Then, by (2.7), (2.8) (for $i = 3$), (2.14), (2.23), and (2.25), we have in turn

$$\begin{aligned}\|x_1 - x^*\| &= \|\psi(x_0, y_0, z_0) - x^*\| \\ &\leq g_3(\|x_0 - x^*\|, \|y_0 - x^*\|, \|z_0 - x^*\|)\|x_0 - x^*\| \\ &\leq g_3(\|x_0 - x^*\|, \|x_0 - x^*\|, \|x_0 - x^*\|)\|x_0 - x^*\| \\ &\leq \|x_0 - x^*\| < \rho^*; \end{aligned} \tag{2.26}$$

so (2.18) holds for for $n = 0$ and $x_1 \in U(x^*, \rho^*)$. If we simply replace x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates, we complete the induction for (2.16)–(2.18). Then, in view of the estimate

$$\|x_{k+1} - x^*\| \leq c\|x_k - x^*\| < \rho^*, \tag{2.27}$$

where $c = g_3(\|x_0 - x^*\|, \|x_0 - x^*\|, \|x_0 - x^*\|) \in [0, 1)$, we deduce that $\lim_{k\longrightarrow\infty} x_k = x^*$ and $x_{k+1} \in U(x^*, \rho^*)$. Let $y^* \in D_1$ be such that $F(y^*) = 0$. Set $T = \int_0^1 F'(x^* + \theta(y^* - x^*))d\theta$. Then, using (2.10) and (2.19), we get

$$\|F'(x^*)^{-1}(T - F'(x^*))\| \leq \int_0^1 w_0(\|y^* - x^*\|)d\theta \leq \int_0^1 w_0(\theta r)d\theta < 1; \tag{2.28}$$

so $T^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$, and from the identity

$$0 = F(y^*) - F(x^*) = T(y^* - x^*). \tag{2.29}$$

We conclude that $x* = y^*$ completing the uniqueness of the solution part and the proof of the theorem. □

Remark 2.2.

1. In view of the estimate

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + w_0(\|x - x^*\|), \end{aligned}$$

   we can set

$$v(t) = 1 + w_0(t)$$

   or $v(t) = 2$.

2. The results, obtained here, can be used for operators F satisfying autonomous differential equations [2, 3, 11] of the form

$$F'(x) = G(F(x)),$$

   where $G : \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous operator. Then, since $F'(x^*) = G(F(p)) = G(0)$, we can apply the results without actually knowing p. For example, let $F(x) = e^x - 1$. Then, we can choose: $G(x) = x + 1$.

3. The local results, obtained here, can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method(GCR) for combined Newton/finite projection methods, and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies in discretization studies [2, 3].

4. If $w_0(t) = L_0 t$ and $w(t) = Lt$, then, the parameter $r_A = \frac{2}{2L_0+L}$ was shown by us to be the convergence radius of Newton's method [2, 3]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \qquad \text{for each } n = 0, 1, 2, \ldots \tag{2.30}$$

   under the conditions (2.11)–(2.14). It follows, from the definitions of radii r, that the convergence radius r of these preceding methods cannot be larger than the convergence radius r_A of the second order Newton's method (2.30). As already noted in [2, 3] r_A is at least as large as the convergence ball given by Rheinboldt [18]

$$r_R = \frac{2}{3L}.$$

   In particular, for $L_0 < L$, we have that

$$r_R < r_A$$

   and

$$\frac{r_R}{r_A} \to \frac{1}{3} \qquad as \ \frac{L_0}{L} \to 0.$$

   That is our convergence ball r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [23].

5. It is worth noticing that the studied methods are not changing when we use the conditions of the preceding theorems instead of the stronger conditions used in [1, 4–27]. Moreover, by the preceding theorems we can compute the computational order of convergence (COC) [24] defined by
$$\xi = \ln\left(\frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}\right) / \ln\left(\frac{\|x_n-x^*\|}{\|x_{n-1}-x^*\|}\right)$$
or the approximate computational order of convergence
$$\xi_1 = \ln\left(\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}\right) / \ln\left(\frac{\|x_n-x_{n-1}\|}{\|x_{n-1}-x_{n-2}\|}\right).$$
This way we obtain in practice the order of convergence without resorting to the computation of higher order derivatives appearing in the method or in the sufficient convergence criteria usually appearing in the Taylor expansions for the proofs of those results.

3. Applications and numerical examples

Application 3.1 Let us specialize method (1.3) by setting $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^i, \alpha = 1$,
$$\varphi(x_k, y_k) = y_k - \bar{\tau}_k F'(x_k)^{-1}F(y_k) \text{ and } \psi(x_k, y_k, z_k) = z_k - \alpha_k F'(z_k)^{-1}F(z_k). \quad (3.1)$$
Then, method (1.3) reduces to method (5.3) in [26] defined by
$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k),\\ z_k &= y_k - \bar{\tau}_k F'(x_k)^{-1}F(y_k), \qquad (3.2)\\ x_{k+1} &= z_k - \alpha_k F'(z_k)^{-1}F(z_k). \end{aligned}$$
It was shown in [26, Theorem 1, Theorem 5] that if operator F is sufficiently many times differentiable and $F'(x)$ is continuous on D, $F'(x^*)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$, then, for x_0 sufficiently close to x^*, method (1.3) converges with order $p \geq 2$ if and only if $\bar{\tau}_k$ and α_k satisfy certain conditions involving hypotheses on higher derivatives for F. Further special choices of $\bar{\tau}_k$ and α_k are given in the following table leading to other pth order methods.

Table 1. Comparison table for pth order methods.

	methods	Order	$\bar{\tau}_k$	α_k
1	Cordero et.al. [4]	4	$2l - F'(x_k)^{-1}F(y_k)$	
2	Sharma [19]	4	$3l - 2F'(x_k)^{-1}[y_k, x_k; F]$	
3	Grau-Senchez et.al. [9]	4	$(2[y_k, x_k; F] - F'(x_k)^{-1})F'(x_k)$	
4	Sharma et.al [21]	4	$r_k = I - \frac{3}{4}(s_k - I) + \frac{9}{8}(s_k - I)^2$ $s_k = F'(x_k)^{-1}F'(y_k)$ $r_k = \frac{1}{2}(-I + \frac{9}{4}F'(y_k)^{-1}F'(x_k) + \frac{3}{4}F'(x_k)^{-1}F'(y_k)$	
5	Gran-Sanchez et. al [9] Xiao et.al [25]	5	$\tau_k = \frac{1}{2}(I + F'(y_k)^{-1}F'(x_k)$	$F'(y_k)^{-1}F'(x_k)$
6	Cardero et.al [12]	5	$2(I - F'(x_k)^{-1}F'(y_k))^{-1}$	$F'(y_k)^{-1}F'(x_k)$
7	Xiao et.al [25] Sharma et.al [20]	5	$y_k = x_k - aF'(x_k)^{-1}F'(x_k)$ $\bar{\tau}_k = ((1 - \frac{1}{2a})I + \frac{1}{2a}F'(x_k)^{-1}F'(y_k))^{-1}$ $a = \frac{1}{2}, (F'(y_k)^{-1}F'(x_k) - I)\theta_k$	$-I + 2(\frac{1}{2a}F'(y_k) + (1 - \frac{1}{2a})F'(x_k))^{-1}F'(x_k)$ $2F'(y_k)^{-1}F'(x_k) - I$
8	Sharma et.al [22]	6	$3I - 2F'(x_k)^{-1}[y_k, x_k; F]$	$3I - 2F'(x_k)^{-1}[y_k, x_k; F]$
9	Xiao et.al [25]	6	$y_k = x_k - aF'(x_k)^{-1}F'(x_k)$ $r_k = \frac{1}{2}(-I + \frac{9}{4}F'(y_k)^{-1})F'(x_k) + \frac{3}{4}F'(x_k))^{-1}F'(y_k)$ $(2[y_k, x_k; F]^{-1} - F'(x_k))^{-1}F'(x_k)$	$(1 - \frac{1}{a})I + \frac{1}{a}F'(y_k)^{-1}F'(x_k)$ $(2[y_k, x_k; F] - F'(x_k))^{-1}F'(x_k)$
10	Grau-Sanchez et.al [9]	6	$(2[y_k, x_k; F] - F'(x_k)^{-1})F'(x_k)$	$2[y_k, x_k; F]^{-1}F'(x_k) - I$
11	Cordero et.al [4]	6	$a = \frac{1}{a}, (F'(x_k) - 2F'(y_k))^{-1}(3F'(x_k)\theta_k^{-1} - 4F'(x_k)$	$(F'(x_k) - 2F'(y_k))^{-1}F(x_k)$

Application 3.2 Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^i, \varphi(x_k, y_k) = \varphi_p(x_k, y_k)$, and ψ as in (3.1) and (3.8), where φ_p denotes the iteration function of pth order. Then, again according to Theorem 6 in [26], the method (1.3) has order of convergence $p+2$ under certain conditions of α_k. As an example, we present the choices given by

$$\alpha_k = \frac{1}{2}(5I - 3F'(x_k)^{-1}F'(y_k)), \tag{3.3}$$

$$\alpha_k = 3I - 2F'(x_k)^{-1}F'(y_k), \tag{3.4}$$

$$\alpha_k = F'(x_k)^{-1}F'(y_k), \tag{3.5}$$

$$\alpha_k = ((1-\frac{1}{\alpha})F'(x_k) + \frac{1}{\alpha}F'(y_k))^{-1}F'(x_k), \tag{3.6}$$

$$\alpha_k = ((1+\frac{1}{\alpha})F'(y_k) - \frac{1}{a}F'(x_k))^{-1}F'(y_k) \tag{3.7}$$

(see [2, 3, 26]). Let us consider the special case of method (1.3) given by

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k), \\ x_{k+1} &= y_k - F'(y_k)^{-1}F(y_k). \end{aligned}$$

Then, we have in turn

$$\begin{aligned} \|y_k - x^*\| &\leq \|F'(x_k)^{-1}\int_0^1 [F'(x^* + \theta(x_k - x^*)) - F'(x^*)](x_k - x^*)d\theta\| \\ &\leq \frac{\int_0^1 w((1-\theta)\|x_k - x^*\|)d\theta\|x_k - x^*\|}{1 - w_0(\|x_k - x^*\|)} \\ &= g_1(\|x_k - x^*\|)\|x_k - x^*\| \end{aligned}$$

and

$$\begin{aligned} \|x_{k+1} - x^*\| &= \|F'(y_k)^{-1}\int_0^1 [F'(x^* + \theta(y_k - x^*)) - F'(x^*))(y_k - x^*)d\theta\| \\ &\quad + \|F'(y_k)^{-1}F'(x^*)\|(\|F'(x^*)^{-1}(F'(x_k) - F'(x^*))\| \\ &\quad + \|F'(x^*)^{-1}(F'(y_k) - F'(x^*))\|)\|F'(x_k)^{-1}F'(x^*)\|\|F'(x_k)^{-1}F(y_k)\| \\ &\leq \frac{\int_0^1 w(\theta\|y_k - x^*\|)d\theta\|y_k - x^*\|}{1 - w_0(\|x_k - x^*\|)} \\ &\quad + \frac{(w_0(\|x_k - x^*\|) + w_0(\|y_k - x^*\|))\int_0^1 v(\theta\|y_k - x^*\|)\|y_k - x^*\|)d\theta}{1 - w_0(\|y_k - x^*\|)} \\ &\leq g_2(\|x_k - x^*\|)\|x_k - x^*\|; \end{aligned}$$

so we can choose $\alpha = 1, g_1(t) = \frac{\int_0^1 w((1-\theta)t)d\theta}{1-w_0(t)}$,

$$g_2(t) = \frac{\int_0^1 w(\theta g_1(t)t)d\theta g_1(t)}{1 - w_0(g_1(t)t)} + \frac{(w_0(t) + w_0(g_1(t)t))\int_0^1 v(\theta g_1(t)t)g_1(t)d\theta}{(1 - w_0(g_1(t)t))(1 - w_0(t))},$$

and $\rho^* = \min\{\rho_1, \rho_2\}$, where ρ_1 and ρ_2 are smallest positive solutions of equations $h_1(t) = 0$ and $h_2(t) = 0$, respectively. Using the above choices, we present the following examples.

Example 3.1. Let us consider a system of differential equations governing the motion of an object and given by

$$F_1'(x) = e^x,\ F_2'(y) = (e-1)y + 1,\ F_3(z) = 1$$

with initial conditions $F_1(0) = F_2(0) = F_3(0) = 0$. Let $F = (F_1, F_, F_3)$. Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3, D = \bar{U}(0,1), p = (0,0,0)^T$. Define function F on D for $w = (x,y,z)^T$ by

$$F(w) = (e^x - 1, \frac{e-1}{2}y^2 + y, z)^T.$$

The Fréchet-derivative is defined by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that, using the (2.11)–(2.14) conditions, we get, for $\alpha = 1$, $w_0(t) = (e-1)t, w(t) = e^{\frac{1}{e-1}}t$, and $v(t) = e^{\frac{1}{e-1}}$. The radii are

$$\rho_1 = 0.3827, \qquad \rho_2 = 0.2523 = \rho^*.$$

Example 3.2. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$, be equipped with the max norm. Let $D = \overline{U}(0,1)$. Define function F on D by

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.8}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta \qquad \text{for each } \xi \in D.$$

Then, we get that $x^* = 0$; so $w_0(t) = 7.5t, w(t) = 15t$, and $v(t) = 2$. Then the radii are

$$\rho_1 = 0.0667, \qquad \rho^* = \rho_2 = 0.0439.$$

Example 3.3. Let $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$ and $D = \bar{U}(0,1)$. Consider the equation

$$x(s) = \int_0^1 T(s,t)(\frac{1}{2}x(t)^{\frac{3}{2}} + \frac{x(t)^2}{8})dt, \tag{3.9}$$

where the kernel T is Green's function defined on the interval $[0,1]\times[0,1]$ by

$$T(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t, \end{cases} \tag{3.10}$$

Define operator $F : C[0,1] \longrightarrow [0,1]$ by

$$F(x(s)) = \int_0^1 T(s,t)(\frac{1}{2}x(t)^{\frac{3}{2}} + \frac{x(t)^2}{8})dt - x(s). \tag{3.11}$$

Then, we have

$$F'(x)\mu(s) = \mu(s) - \int_0^1 T(s,t)(\frac{3}{4}x(t)^{\frac{1}{2}} + \frac{x(t)}{4})\mu(t)dt. \tag{3.12}$$

Notice that $x^*(s) = 0$ is a solution of $F(x(s)) = 0$. Using (3.10), we obtain

$$\|\int_0^1 T(s,t)dt\| \leq \frac{1}{8}. \tag{3.13}$$

Then, by (3.12) and (3.13), we have that

$$\|F'(x) - F'(y)\| \leq \frac{1}{32}(3\|x-y\|^{\frac{1}{2}} + \|x-y\|). \tag{3.14}$$

We have $w_0(t) = w(t) = \frac{1}{32}(3t^{1/2} + t)$ and $v(t) = 1 + w_0(t)$. Then the radii are

$$\rho_1 = 19.4772, \qquad \rho_2 = 0.3889;$$

so we choose $\rho^* = 1$ since, by (2.15), $\bar{U}(x^*, \rho^*) \subset D$ [4–27].

In view of (3.14), earlier results requiring hypotheses on the second Fréchet derivative or higher cannot be used to solve equation $F(x(s)) = 0$.

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