



UNIQUENESS OF MEROMORPHIC FUNCTIONS WITH REGARD TO MULTIPLICITY

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ABSTRACT. In this paper, we investigate the uniqueness problem on meromorphic functions concerning differential polynomials sharing one value. A uniqueness result which related to multiplicity of meromorphic function is proved in this paper. By using the notion of multiplicity our results will generalise and improve the result due to Chao Meng [10].

1. INTRODUCTION AND MAIN RESULTS

Let f be a non-constant meromorphic function in the whole complex plane. We shall use the following standard notations of the value distribution theory: $T(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $m(r, f)$, (see [4], [15], [17]). The notation $S(r, f)$ is defined to be any quantity satisfying $S(r, f) = o(T(r, f))$ as $r \rightarrow \infty, r \notin E$, where E is a set of positive real number of finite linear measure, not necessarily the same at each occurrence. The notations $T(r)$ and $S(r)$ are defined respectively by

$$T(r) = \max \{T(r, f), T(r, g)\}, \quad S(r) = o(T(r)) \text{ as } r \rightarrow \infty, r \notin E,$$

for any two nonconstant meromorphic functions f and g . A meromorphic function a is called a small function with respect to f provided that $T(r, a) = S(r, f)$. Moreover, $GCD(n_1, n_2, \dots, n_k)$ denotes the greatest common divisor of positive integers n_1, n_2, \dots, n_k .

Let f and g be two nonconstant meromorphic functions, and let $a \in \mathbb{C}$. We say that f and g share the value a CM(counting multiplicities), provided that $f - a$ and $g - a$ have the same zeros with the same multiplicities. If $f - a$ and $g - a$ have

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the same zeros, then we say that f and g share a IM (ignoring multiplicities). Similarly, we immediately get the definitions of f and g share a IM (or CM), where a is a small function of f and g . In addition, we also need the following notation, for any $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$,

$$\Theta(a, f) = 1 - \limsup_{r \rightarrow \infty} \frac{\overline{N}(r, a; f)}{T(r, f)}.$$

For a complex number $a \in \mathbb{C} \cup \{\infty\}$ and k a positive integer. We denote by $E_k(a, f)$ the set of all a -points of f where an a -point with multiplicity m is counted m times if $m \leq k$ and $k + 1$ times if $m > k$. For a complex number $a \in \mathbb{C} \cup \{\infty\}$, such that $E_k(a, f) = E_k(a, g)$, then we say that f and g share the value a with weight k .

We write f, g share (a, k) to mean that f, g share the value a with weight k . Clearly if f, g share (a, k) then f, g share (a, p) for all integer p , $0 \leq p \leq k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively. We call f and g share (z, k) if $f - z$ and $g - z$ share $(0, k)$.

In 1997, C.C. Yang and X. H. Hua proved the following result.

Theorem A [18]. Let f and g be two non-constant meromorphic functions, $n \geq 11$ an integer and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value a CM, then either $f = dg$ for some $(n+1)$ th root of unity d or $g(z) = c_1 e^{cz}$ and $f(z) = c_2 e^{-cz}$, where c, c_1 and c_2 are constants and satisfy $(c_1 c_2)^{n+1} c^2 = -a^2$.

In 2001, M.L. Fang and W. Hong obtained the following result.

Theorem B [3]. Let f and g be two transcendental entire functions, $n \geq 11$ an integer. If $f^n (f - 1) f'$ and $g^n (g - 1) g'$ share the value 1 CM, then $f \equiv g$.

In 2004, W.C. Lin and H.X. Yi extended the above theorem in view of the fixed-point. They proved the following result.

Theorem C [7]. Let f and g be two transcendental meromorphic functions, $n \geq 13$ an integer. If $f^n (f - 1)^2 f'$ and $g^n (g - 1)^2 g'$ share z CM, then $f \equiv g$.

In 2006, I. Lahiri and R. Pal proved the following result.

Theorem D [6]. Let f and g be two non-constant meromorphic functions, $n \geq 14$ be an integer. If $E_3(1, f^n (f^3 - 1) f') = E_3(1, g^n (g^3 - 1) g')$, then $f \equiv g$.

In 2008, Chao Meng relaxed the nature of fixed point to IM and proved the following result.

Theorem E [8]. Let f and g be two transcendental meromorphic functions, $n \geq 28$ an integer. If $f^n (f - 1)^2 f'$ and $g^n (g - 1)^2 g'$ share z IM, then $f \equiv g$.

In 2009, Chao Meng relaxed nature of sharing value in the above theorem and proved the following result.

Theorem F [9]. Let f and g be two non-constant meromorphic functions such that $f^n (f^3 - 1) f'$ and $g^n (g^3 - 1) g'$ share $(1, l)$ where n be a positive integer such that $(n + 1)$ is not divisible by 3. If

(i) $l = 2$ and $n \geq 14$; (ii) $l = 1$ and $n \geq 17$; (iii) $l = 0$ and $n \geq 35$ then $f \equiv g$.

In 2015, Chao Meng proved the following result.

Theorem G [10]. Let f and g be two non-constant meromorphic functions,

$n \geq 12$ a positive integer. If $f^n(f^3 - 1)f'$ and $g^n(g^3 - 1)g'$ share $(1, 2)$, f and g share ∞ IM, then $f \equiv g$.

In this paper, using notion of multiplicity, we prove the following Theorems.

Theorem 1. Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast s , where s is a positive integer. Let n, m be positive integers with $s(n - m - 3) \geq 6$ and let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_1 z + a_0$, where $a_0 \neq 0, a_1, a_2, \dots, a_{m-1}, a_m \neq 0$ are complex constants. If $f^n P(f)f'$ and $g^n P(g)g'$ share $(1, 2)$, f and g share ∞ IM, then one of the following two cases hold:

- (i) $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = GCD(n + m + 1, \dots, n + m + 1 - i, \dots, n + 1)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \dots, m$;
- (ii) f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(w_1, w_2) = w_1^{n+1} \left(\frac{a_m w_1^n}{n+m+1} + \frac{a_{m-1} w_1^{n-1}}{n+m} + \dots + \frac{a_0}{n+1} \right) - w_2^{n+1} \left(\frac{a_m w_2^n}{n+m+1} + \frac{a_{m-1} w_2^{n-1}}{n+m} + \dots + \frac{a_0}{n+1} \right)$.

Theorem 2. Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast s , where s is a positive integer. Let n, m be positive integers with $s(n - m) > 3k + 4 + (k + 2)s$, $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$ and let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \dots + a_1 z + a_0$, where $a_0 \neq 0, a_1, a_2, \dots, a_{m-1}, a_m \neq 0$ are complex constants. If $[f^n P(f)]^{(k)}$ and $[g^n P(g)]^{(k)}$ share $(1, 2)$, f and g share ∞ IM, then one of the following two cases hold:

- (i) $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = GCD(n + m, \dots, n + m - i, \dots, n + 1, n)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \dots, m$;
- (ii) f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(f, g) = f^n P(f) - g^n P(g)$.

Remark.

(1) We set $P(z) = (z - 1)^m$. Then with $a_m = 1, a_0 = -1$ and under the condition (ii) of Theorem 1, Theorem 1 reduces to Theorem G, if $m = 3, s = 1$.

(2) In Theorem 1, giving specific values for s in Theorem 1, we get the following interesting cases

- i) If $s = 1$, then $n \geq m + 9$.
- ii) If $s = 2$, then $n \geq m + 6$.
- iii) If $s = 3$, then $n \geq m + 5$.

We conclude that if f and g have zeros and poles of higher order multiplicity, then we can reduce the values of n .

(3) In order to discuss the case of sharing IM (ignoring multiplicity), we have to replace the sharing $(1, 2)$ in the above Theorems 1 and 2 by $(1, 0)$. Then we obtain the same result for the value of n higher than the above mentioned Theorems 1 and 2.

The following example shows that f and g share ∞ IM, in Theorems 1 and 2 can not be removed.

Example. Let $P(z) = (z - 1)^6(z + 1)^6 z^{11}$, $f(z) = \sin z$, $g(z) = \cos z$ and $k = 0, s = 1$. It is easy to see that $n > m + 6$ and $P(f(z))f^n(z) = P(g(z))g^n(z)$.

Therefore $P(f(z))f^n(z)$ and $P(g(z))g^n(z)$ share 1 CM. It is also clear that though f and g satisfy $R(f, g) = 0$, where $R(w_1, w_2) = P(w_1)w_1(z) - P(w_2)w_2(z)$, we have $f \not\equiv tg$ for a constant t satisfying $t^m = 1$, where $m \in \mathbb{Z}^+$.

2. SOME LEMMAS

For the proof of our main results, we need the following lemmas.

Lemma 2.1. [1] *If F and G share $(1, 2)$ and (∞, k) where $0 \leq k \leq \infty$, then one of the following cases hold:*

(1) $T(r, F) \leq N_2(r, \frac{1}{F}) + N_2(r, \frac{1}{G}) + \bar{N}(r, F) + \bar{N}(r, G) + \bar{N}_*(r, \infty; F, G) + S(r, F) + S(r, G)$, the same inequality holds for $T(r, G)$;

(2) $F \equiv G$;

(3) $FG \equiv 1$.

Lemma 2.2. [19] *Suppose $f(z)$ is a non-constant meromorphic function in the complex plane and k is a positive integer. Then*

$$N\left(r, \frac{1}{f^{(k)}}\right) \leq N\left(r, \frac{1}{f}\right) + k\bar{N}(r, f) + S(r, f).$$

Lemma 2.3. [20] *Let f be a non-constant meromorphic function and p, k be positive integers. Then*

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq T(r, f^{(k)}) - T(r, f) + N_{p+k}\left(r, \frac{1}{f}\right), \quad (2.1)$$

$$N_p\left(r, \frac{1}{f^{(k)}}\right) \leq k\bar{N}(r, f) + N_{p+k}\left(r, \frac{1}{f}\right) + S(r, f). \quad (2.2)$$

Lemma 2.4. [16] *Let f be a non-constant meromorphic function and let a_1, a_2, \dots, a_n ($\neq 0$) be finite complex numbers. Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + \dots + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2.5. [14] *Let f and g be two non constant meromorphic functions. $f^n P(f) f' g^n P(g) g' \neq 1$, where $n + m$ (≥ 6) is a positive integer.*

Lemma 2.6. [5] *Let f and g be two non constant meromorphic functions such that $\Theta(\infty, f) + \Theta(\infty, g) > \frac{4}{n}$ for all integers $n \geq 3$. Then $f^n(af + b) = g^n(ag + b)$ implies $f = g$, where a and b are two finite nonzero complex constants.*

Lemma 2.7. *Let f and g be two non-constant meromorphic functions whose zeros and poles are of multiplicities at least s , where s is a positive integer and let n, k be positive integers. Let $F = [f^n P(f)]^{(k)}$ and $G = [g^n P(g)]^{(k)}$, where $p(z)$ be defined as in Theorem 1. If there exists two non zero constants b_1 and b_2 such that $\bar{N}\left(r, \frac{1}{F}\right) = \bar{N}\left(r, \frac{1}{G-b_1}\right)$ and $\bar{N}\left(r, \frac{1}{G}\right) = \bar{N}\left(r, \frac{1}{F-b_2}\right)$, then $(n-m)s \leq 3k+3$.*

Proof. By the second fundamental theorem of Nevanlinna's theory,

$$\begin{aligned} T(r, F) &\leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}(r, F) + \bar{N}\left(r, \frac{1}{F-b_2}\right) + S(r, F) \\ &\leq \bar{N}\left(r, \frac{1}{F}\right) + \bar{N}(r, F) + \bar{N}\left(r, \frac{1}{G}\right) + S(r, F). \end{aligned} \quad (2.3)$$

Combining (2.1), (2.2), (2.3), and Lemma 2.4, we get

$$\begin{aligned}
(n+m)T(r, f) &\leq T(r, F) - \bar{N}\left(r, \frac{1}{F}\right) + N_{k+1}\left(r, \frac{1}{f^n P(f)}\right) + S(r, f) \\
&\leq \bar{N}\left(r, \frac{1}{G}\right) + \bar{N}(r, f) + N_{k+1}\left(r, \frac{1}{f^n P(f)}\right) + S(r, f) \\
&\leq N_{k+1}\left(r, \frac{1}{f^n P(f)}\right) + N_{k+1}\left(r, \frac{1}{g^n P(g)}\right) + k\bar{N}(r, g) \\
&\quad + \bar{N}(r, f) + S(r, f) + S(r, g) \\
&\leq \left(\frac{k+2}{s} + m\right)T(r, f) + \left(\frac{2k+1}{s} + m\right)T(r, g) \\
&\quad + S(r, f) + S(r, g) \\
&\leq \left(\frac{3k+3}{s} + 2m\right)T(r) + S(r).
\end{aligned} \tag{2.4}$$

Similarly, for the case of g ,

$$(n+m)T(r, g) \leq \left(\frac{3k+3}{s} + 2m\right)T(r) + S(r). \tag{2.5}$$

It follows from (2.4) and (2.5) that

$$\left(n - m - \left(\frac{3k+3}{s}\right)\right)T(r) \leq S(r)$$

which gives $(n-m)s \leq 3k+3$. This completes the proof. \square

Lemma 2.8. [13] *Let f and g be two non-constant meromorphic functions, and let $n(\geq 1)$, $k(\geq 1)$ and $m(\geq 1)$ be a integers. Then $[f^n P(f)]^{(k)}[g^n P(g)]^{(k)} \neq 1$.*

3. PROOF OF THE THEOREMS

Proof of Theorem 1.

Let

$$F = f^n P(f) f', \quad G = g^n P(g) g' \tag{3.1}$$

$$\text{and} \quad F^* = \frac{a_m}{m+n+1} f^{m+n+1} + \frac{a_{m-1}}{m+n} f^{m+n} + \dots + \frac{a_0}{n+1} f^{n+1} \tag{3.2}$$

$$G^* = \frac{a_m}{m+n+1} g^{m+n+1} + \frac{a_{m-1}}{m+n} g^{m+n} + \dots + \frac{a_0}{n+1} g^{n+1}. \tag{3.3}$$

Thus we obtain that F and G share (1, 2). If possible, let the case 1 of Lemma 2.1 i.e.,

$$\begin{aligned}
T(r, F) &\leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \bar{N}(r, F) + \bar{N}(r, G) + \bar{N}_*(r, \infty; F, G) \\
&\quad + S(r, F) + S(r, G).
\end{aligned} \tag{3.4}$$

Moreover, by Lemma 2.4 we have

$$T(r, F^*) = (m + n + 1)T(r, f) + S(r, f), \quad (3.5)$$

$$T(r, G^*) = (m + n + 1)T(r, g) + S(r, g). \quad (3.6)$$

Since $(F^*)' = F$, we deduce

$$m \left(r, \frac{1}{F^*} \right) = m \left(r, \frac{1}{F} \right) + S(r, f). \quad (3.7)$$

By Neenanlinna's first fundamental theorem, we get

$$\begin{aligned} T(r, F^*) &\leq \bar{N}(r, F) + N \left(r, \frac{1}{F^*} \right) - N \left(r, \frac{1}{F} \right) + S(r, f) \\ &\leq T(r, F) + N \left(r, \frac{1}{f} \right) + \sum_{i=1}^m N \left(r, \frac{1}{f - b_i} \right) - \sum_{i=1}^m N \left(r, \frac{1}{f - c_i} \right) \\ &\quad - N \left(r, \frac{1}{f'} \right) + S(r, f), \end{aligned} \quad (3.8)$$

where b_1, b_2, \dots, b_m are roots of algebraic equation

$$\frac{a_m}{m + n + 1} z^m + \frac{a_{m-1}}{m + n} z^{m-1} + \dots + \frac{a_0}{n + 1} = 0,$$

and c_1, c_2, \dots, c_m are roots of algebraic equation $a_m z^m + a_{m-1} z^{m-1} + \dots + a_0 = 0$.

It follows from (3.4) and (3.8) that

$$\begin{aligned} T(r, F^*) &\leq N_2 \left(r, \frac{1}{F} \right) + N_2 \left(r, \frac{1}{G} \right) + \bar{N}(r, F) + \bar{N}(r, G) + \bar{N}_*(r, \infty; F, G) \\ &\quad + N \left(r, \frac{1}{f} \right) + \sum_{i=1}^m N \left(r, \frac{1}{f - b_i} \right) - \sum_{i=1}^m N \left(r, \frac{1}{f - c_i} \right) - N \left(r, \frac{1}{f'} \right) \\ &\quad + S(r, f). \end{aligned} \quad (3.9)$$

It follows from (3.1) that

$$\begin{aligned} N_2 \left(r, \frac{1}{F} \right) + \bar{N}(r, F) &\leq 2\bar{N} \left(r, \frac{1}{f} \right) + N \left(r, \frac{1}{f - c_1} \right) + \dots + N \left(r, \frac{1}{f - c_m} \right) \\ &\quad + N \left(r, \frac{1}{f'} \right) + \bar{N}(r, f). \end{aligned} \quad (3.10)$$

$$\begin{aligned} N_2 \left(r, \frac{1}{G} \right) + \bar{N}(r, G) &\leq 2\bar{N} \left(r, \frac{1}{g} \right) + N \left(r, \frac{1}{g - c_1} \right) + \dots + N \left(r, \frac{1}{g - c_m} \right) \\ &\quad + N \left(r, \frac{1}{g'} \right) + \bar{N}(r, g). \end{aligned} \quad (3.11)$$

From (3.9), (3.10) and (3.11) we obtain

$$\begin{aligned} T(r, F^*) &\leq 2\bar{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-b_1}\right) + \dots + N\left(r, \frac{1}{f-b_m}\right) + N\left(r, \frac{1}{f}\right) \\ &\quad + 2\bar{N}\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g-c_1}\right) + \dots + N\left(r, \frac{1}{g-c_m}\right) + N\left(r, \frac{1}{g}\right) \\ &\quad + \bar{N}_*(r, \infty; F, G) + \bar{N}(r, f) + \bar{N}(r, g) + S(r, f) + S(r, g). \end{aligned} \quad (3.12)$$

By Lemma 2.2 we have,

$$N\left(r, \frac{1}{g}\right) \leq N\left(r, \frac{1}{g}\right) + \bar{N}(r, g) + S(r, f). \quad (3.13)$$

Also, we have

$$\bar{N}(r, f) + \bar{N}(r, g) + \bar{N}_*(r, \infty; F, G) \leq N(r, f) + N(r, g). \quad (3.14)$$

By our assumption, zeros and poles of f and g are of multiplicities atleast s , we have

$$\bar{N}(r, g) \leq \frac{1}{s}N(r, g) \leq \frac{1}{s}T(r, g), \quad (3.15)$$

$$\bar{N}\left(r, \frac{1}{g}\right) \leq \frac{1}{s}N\left(r, \frac{1}{g}\right) \leq \frac{1}{s}T(r, g). \quad (3.16)$$

We deduce from (3.12) - (3.16) that

$$\begin{aligned} (n+m+1)T(r, f) &\leq \left(\frac{2}{s} + m + 2\right)T(r, f) + \left(\frac{3}{s} + m + 2\right)T(r, g) \\ &\quad + S(r, f) + S(r, g), \end{aligned} \quad (3.17)$$

$$\left(n - \frac{2}{s} - 1\right)T(r, f) \leq \left(\frac{3}{s} + m + 2\right)T(r, g) + S(r, f) + S(r, g). \quad (3.18)$$

Similarly,

$$\left(n - \frac{2}{s} - 1\right)T(r, g) \leq \left(\frac{3}{s} + m + 2\right)T(r, f) + S(r, f) + S(r, g). \quad (3.19)$$

From (3.17) and (3.19), we deduce that $(n-m-3)s \leq 5$, which contradicts the assumption $(n-m-3)s \geq 6$.

Case 2: Suppose $FG \equiv 1$, by Lemma 2.5, we get a contradiction.

Case 3: If $F \equiv G$, that is

$$F^* = G^* + c$$

where c is a constant, then it follows that

$$T(r, f) = T(r, g) + S(r, f). \quad (3.20)$$

Suppose that $c \neq 0$, by (3.2), (3.3), (3.5), (3.15), (3.16), (3.20), the second fundamental theorem, and lemma 2.4 we have

$$\begin{aligned}
T(r, G^*) &\leq \bar{N}\left(r, \frac{1}{G^*}\right) + \bar{N}\left(r, \frac{1}{G^* + c}\right) + \bar{N}(r, G^*) + S(r, g) \\
(n+m+1)T(r, g) &\leq \bar{N}\left(r, \frac{1}{G^*}\right) + \bar{N}\left(r, \frac{1}{F^*}\right) + \bar{N}(r, G^*) + S(r, g) \\
&\leq \bar{N}\left(r, \frac{1}{g}\right) + \bar{N}\left(r, \frac{1}{g^m + \dots + \frac{m+n+1}{n+1} \frac{a_0}{a_m}}\right) + \bar{N}\left(r, \frac{1}{f}\right) \\
&\quad + \bar{N}\left(r, \frac{1}{f^m + \dots + \frac{m+n+1}{n+1} \frac{a_0}{a_m}}\right) + \bar{N}(r, g) + S(r, g) \\
&\leq \left(\frac{3}{s} + 2m\right) T(r, f) + S(r, f) + S(r, g)
\end{aligned} \tag{3.21}$$

which contradicts our assumption $(n-m-3)s \geq 6$. Therefore $F^* = G^*$ that is,

$$\begin{aligned}
&f^{n+1} \left(\frac{a_m f^m}{m+n+1} + \frac{a_{m-1} f^{m-1}}{m+n} + \dots + \frac{a_0}{n+1} \right) \\
&= g^{n+1} \left(\frac{a_m g^m}{m+n+1} + \frac{a_{m-1} g^{m-1}}{m+n} + \dots + \frac{a_0}{n+1} \right).
\end{aligned} \tag{3.22}$$

Let $h = \frac{f}{g}$. If h is a constant, then substituting $f = gh$ into (3.22) we deduce,

$$\frac{a_m g^{m+n+1} (h^{m+n+1} - 1)}{m+n+1} + \frac{a_{m-1} g^{m+n} (h^{m+n} - 1)}{m+n} + \dots + \frac{a_0 g^{n+1} (h^{n+1} - 1)}{n+1} = 0 \tag{3.23}$$

which implies $h^d = 1$, where $d = (n+m+1, n+m, \dots, n+m+1-i, \dots, n+1)$, $a_{m-i} \neq 0$, for some $i = 0, 1, \dots, m$.

Thus $f = tg$ for a constant t , such that $t^d = 1$, where $d = (n+m+1, n+m, \dots, n+m+1-i, \dots, n+1)$, $a_{m-i} \neq 0$, for some $i = 0, 1, \dots, m$.

If h is not a constant, then by (3.23) f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where

$$\begin{aligned}
R(w_1, w_2) &= w_1^{n+1} \left(\frac{a_m w_1^m}{n+m+1} + \frac{a_{m-1} w_1^{m-1}}{n+m} + \dots + \frac{a_0}{n+1} \right) \\
&\quad - w_2^{n+1} \left(\frac{a_m w_2^m}{n+m+1} + \frac{a_{m-1} w_2^{m-1}}{n+m} + \dots + \frac{a_0}{n+1} \right).
\end{aligned}$$

This completes the proof of Theorem 1.

Proof of Theorem 2.

Let

$$F = [f^n P(f)]^{(k)}, \quad G = [g^n P(g)]^{(k)} \tag{3.24}$$

$$\begin{aligned}
\text{and } F^* &= \frac{a_m(n+m)!}{(n+m-k+1)!} f^{n+m-k+1} + \frac{a_{m-1}(n+m-1)!}{(n+m-k)!} f^{n+m-k} \\
&+ \dots + \frac{a_0 n!}{(n-k+1)!} f^{n-k+1}
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
G^* &= \frac{a_m(n+m)!}{(n+m-k+1)!} g^{n+m-k+1} + \frac{a_{m-1}(n+m-1)!}{(n+m-k)!} g^{n+m-k} \\
&+ \dots + \frac{a_0 n!}{(n-k+1)!} g^{n-k+1}
\end{aligned} \tag{3.26}$$

Thus we obtain that F and G share $(1, 2)$. If possible, let the case 1 of Lemma 2.1 i.e.,

$$\begin{aligned}
T(r, F) &\leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \bar{N}(r, F) + \bar{N}(r, G) + \bar{N}_*(r, \infty; F, G) \\
&+ S(r, F) + S(r, G).
\end{aligned} \tag{3.27}$$

Moreover, by Lemma 2.4, we have

$$T(r, F^*) = (n+m-k+1)T(r, f) + S(r, f), \tag{3.28}$$

$$T(r, G^*) = (n+m-k+1)T(r, g) + S(r, g). \tag{3.29}$$

Since $(F^*)' = F$, we deduce

$$m\left(r, \frac{1}{F^*}\right) = m\left(r, \frac{1}{F}\right) + S(r, f). \tag{3.30}$$

By Nevanlinna's first fundamental theorem, we get

$$\begin{aligned}
T(r, F^*) &\leq \bar{N}(r, F) + N\left(r, \frac{1}{F^*}\right) - N\left(r, \frac{1}{F}\right) + S(r, f) \\
&\leq T(r, F) + N\left(r, \frac{1}{f}\right) + \sum_{i=1}^m N\left(r, \frac{1}{f-b_i}\right) - \sum_{i=1}^m N\left(r, \frac{1}{f-c_i}\right) \\
&\quad - k\bar{N}(r, f) + S(r, f),
\end{aligned} \tag{3.31}$$

where b_1, b_2, \dots, b_m are roots of algebraic equation

$$\frac{a_m(n+m)!}{(n+m-k+1)!} z^m + \frac{a_{m-1}(n+m-1)!}{(n+m-k)!} z^{m-1} + \dots + \frac{a_0 n!}{(n-k+1)!} = 0,$$

and c_1, c_2, \dots, c_m are roots of algebraic equation $a_m z^m + a_{m-1} z^{m-1} + \dots + a_0 = 0$.

It follows from (3.27) and (3.31) that

$$\begin{aligned} T(r, F^*) &\leq N_2\left(r, \frac{1}{F}\right) + N_2\left(r, \frac{1}{G}\right) + \bar{N}(r, F) + \bar{N}(r, G) + \bar{N}_*(r, \infty; F, G) \\ &\quad + N\left(r, \frac{1}{f}\right) + \sum_{i=1}^m N\left(r, \frac{1}{f-b_i}\right) - \sum_{i=1}^m N\left(r, \frac{1}{f-c_i}\right) - k\bar{N}(r, f) \\ &\quad + S(r, f). \end{aligned} \tag{3.32}$$

It follows from (3.24) that

$$\begin{aligned} N_2\left(r, \frac{1}{F}\right) + \bar{N}(r, F) &= N_2\left(r, \frac{1}{[f^n P(f)]^{(k)}}\right) + \bar{N}(r, F) \\ &\leq (k+2)\bar{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-c_1}\right) + \dots + N\left(r, \frac{1}{f-c_m}\right) \\ &\quad + (k+1)\bar{N}(r, f), \end{aligned} \tag{3.33}$$

$$\begin{aligned} N_2\left(r, \frac{1}{G}\right) + \bar{N}(r, G) &= N_2\left(r, \frac{1}{[g^n P(g)]^{(k)}}\right) + \bar{N}(r, G) \\ &\leq (k+2)\bar{N}\left(r, \frac{1}{g}\right) + N\left(r, \frac{1}{g-c_1}\right) + \dots + N\left(r, \frac{1}{g-c_m}\right) \\ &\quad + (k+1)\bar{N}(r, g). \end{aligned} \tag{3.34}$$

From (3.32), (3.33) and (3.34) we obtain

$$\begin{aligned} T(r, F^*) &\leq (k+2)\bar{N}\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f-b_1}\right) + \dots + N\left(r, \frac{1}{f-b_m}\right) \\ &\quad + k\bar{N}(r, f) + \bar{N}(r, f) + (k+2)\bar{N}\left(r, \frac{1}{g}\right) + k\bar{N}(r, g) + \bar{N}(r, g) \\ &\quad + N\left(r, \frac{1}{g-c_1}\right) + \dots + N\left(r, \frac{1}{g-c_m}\right) + \bar{N}_*(r, \infty; F, G) + N\left(r, \frac{1}{f}\right) \\ &\quad - k\bar{N}(r, f) + S(r, f) + S(r, g). \end{aligned} \tag{3.35}$$

Also, we have

$$\bar{N}(r, f) + \bar{N}(r, g) + \bar{N}_*(r, \infty; F, G) \leq N(r, f) + N(r, g). \tag{3.36}$$

By our assumption, zeros and poles of f and g are of multiplicities atleast s , we have

$$\bar{N}(r, g) \leq \frac{1}{s}N(r, g) \leq \frac{1}{s}T(r, g) \tag{3.37}$$

$$\bar{N}\left(r, \frac{1}{g}\right) \leq \frac{1}{s}N\left(r, \frac{1}{g}\right) \leq \frac{1}{s}T(r, g). \tag{3.38}$$

We deduce from (3.35) -(3.38) that

$$\begin{aligned} (n+m-k+1)T(r, f) &\leq \left(\frac{k+2}{s} + 2 + m\right)T(r, f) + \left(\frac{2k+2}{s} + 1 + m\right)T(r, g) \\ &\quad + S(r, f) + S(r, g) \\ \left(n-1-k-\frac{(k+2)}{s}\right)T(r, f) &\leq \left(\frac{2k+2}{s} + 1 + m\right)T(r, g) + S(r, f) + S(r, g). \end{aligned} \quad (3.39)$$

Similarly,

$$\left(n-1-k-\frac{(k+2)}{s}\right)T(r, g) \leq \left(\frac{2k+2}{s} + 1 + m\right)T(r, f) + S(r, f) + S(r, g). \quad (3.40)$$

From (3.39) and (3.40), we deduce that $(n-m)s \leq 3k+4+(k+2)s$, which contradicts the assumption $(n-m)s > 3k+4+(k+2)s$.

Case 2: Suppose $FG \equiv 1$, by Lemma 2.8, we get a contradiction.

Case 3: If $F \equiv G$, this implies that

$$[f^n P(f)]^{(k)} = [g^n P(g)]^{(k)}. \quad (3.41)$$

Integrating for (3.41), we have

$$[f^n P(f)]^{(k-1)} = [g^n P(g)]^{(k-1)} + b_{k-1}, \quad (3.42)$$

where b_{k-1} is constant. If $b_{k-1} \neq 0$, we obtain $(n-m)s \leq 3k+3 < 3k+4+(k+2)s$ by Lemma 2.7. This is a contradiction with our assumption that $(n-m)s > 3k+4+(k+2)s$. Thus $b_{k-1} = 0$. By repeating $k-$ times,

$$f^n P(f) = g^n P(g). \quad (3.43)$$

If $m = 1$ in (3.43), then $f = g$ by Lemma 2.6. Suppose that $m \geq 2$ and $h = \frac{f}{g}$. If h is a constant, putting $f = gh$ in (3.43), we get

$$a_m g^{n+m} (h^{n+m} - 1) + a_{m-1} g^{n+m-1} (h^{n+m-1} - 1) + \dots + a_0 g^n (h^n - 1) = 0, \quad (3.44)$$

which implies $h^d = 1$, where $d = \text{GCD}(n+m, \dots, n+m-i, \dots, n+1, n)$. Hence $f \equiv tg$ for a constant t such that $t^d = 1$, $d = \text{GCD}(n+m, \dots, n+m-i, \dots, n+1, n)$, $i = 0, 1, \dots, m$.

If h is not a constant, then we can see that f and g satisfy the algebraic equation $R(f, g) = 0$, by (3.44), where $R(f, g) = f^n P(f) - g^n P(g)$. This completes the proof of Theorem 2.

4. Open problems.

Following questions posed from the results:

Question 4.1. Can n in Theorems 1 – 2 be still reduced ?

Question 4.2. Is it possible to replace the non-constant meromorphic functions by non-constant entire functions ?

Question 4.3. What can be said about if the sharing one value is replaced by a

small function?

Question 4.4. Can $(1, 2)$ shared value be replaced by $(1, l)$ ($l \geq 0$) shared value in Theorems 1 – 2 ?.

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