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MORE ON CONVERGENCE THEORY OF PROPER MULTISPLITTINGS

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Abstract. In this paper, we first prove a few comparison results between two proper weak regular splittings which are useful in getting the iterative solution of a large class of rectangular (square singular) linear system of equations $Ax = b$, in a faster way. We then derive convergence and comparison results for proper weak regular multisplittings.

1. Introduction

Berman and Plemmons [3] introduced the notion of proper splitting for rectangular/square singular matrices in order to find the least squares solution of minimum norm of a rectangular system of linear equations of the form

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, which we recall next. A splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called a *proper splitting* if $R(U) = R(A)$ and $N(U) = N(A)$, where $R(A)$ and $N(A)$ denote the range space and the null space of A, respectively. Then, the same authors proved that the iterative scheme:

$$x^{k+1} = U^\dagger V x^k + U^\dagger b,\ \ k = 0, 1, 2, \dots \tag{1.2}$$

converges to $A^\dagger b$, the least squares solution of minimum norm for any initial vector x^0 if and only if the spectral radius of $U^\dagger V$ is less than 1 (see Corollary 1, [3]). The above iterative scheme is said to be *convergent* if the spectral radius

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of the iteration matrix $U^\dagger V$ is strictly less than 1. The advantage of the iterative technique for solving the rectangular system of linear equations ($Ax = b$) is that it avoids the use of the normal system $A^TAx = A^Tb$, where A^TA is frequently ill-conditioned and influenced greatly by roundoff errors (see [12]). Such systems appear in deconvolution problems with a smooth kernel. Square singular linear systems also appear in problems like the finite difference representation of Neumann problems.

The authors of [3] obtained several convergence criteria for (1.2). In the recent years, several convergence and comparison results for different subclasses of proper splittings have been proved by many authors such as Baliarsingh and Mishra [1], Climent *et al.* [6], Jena *et al.* [13], Mishra [15]. To get faster convergence, Climent *et al.* [8] introduced the notion of proper multisplittings and obtained convergence criteria by extending the work of O'leary and White [16] to rectangular matrices. This article further continues to investigate the comparisons of the rate of convergence of two iterative schemes in order to get the desired solution in less time.

The paper is organized as follows. The next section contains notation, definitions and preliminary tools. In Section 3, we prove our main results. First we prove a couple of comparison results between two proper weak regular splittings and then we discuss a few applications of theory of proper weak regular splittings to multisplitting theory of rectangular matrices.

2. Preliminary notions and results

The notation $\mathbb{R}^{m\times n}$ represents the set of all real matrices of order $m \times n$. We denote the transpose of a matrix $A \in \mathbb{R}^{m\times n}$ by A^T. Let L and M be complementary subspaces of $\mathbb{R}^n$, and $P_{L,M}$ be a projection onto L along M. Then $P_{L,M}A = A$ if and only if $R(A) \subseteq L$, and $AP_{L,M} = A$ if and only if $N(A) \supseteq M$. In the case of $L \perp M$, $P_{L,M}$ will be denoted by P_L for notational simplicity. The *spectral radius* of a matrix $A \in \mathbb{R}^{n\times n}$ is denoted by $\rho(A)$, is defined by $\rho(A) = \max\limits_{1\leq i\leq n} |\lambda_i|$, where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A. Let A and B be two matrices of appropriate order such that the products AB and BA are defined. Then $\rho(AB) = \rho(BA)$. Let $A \in \mathbb{R}^{m\times n}$, by $A \geq 0$ we denote the matrix whose entries are non-negative. Let $B, C \in \mathbb{R}^{m\times n}$. We write $B \geq C$ if $B - C \geq 0$. The same notation and nomenclature are also used for vectors. For $A \in \mathbb{R}^{m\times n}$, the unique matrix $Z \in \mathbb{R}^{n\times m}$ satisfying the following four equations known as Penrose equations: $AZA = A$, $ZAZ = Z$, $(AZ)^T = AZ$ and $(ZA)^T = ZA$ is called the *Moore-Penrose inverse* of A. It always exists, and is denoted by $A^\dagger$. The following properties of $A^\dagger$ will be frequently used in this paper: $R(A^T) = R(A^\dagger)$; $N(A^T) = N(A^\dagger)$; $AA^\dagger = P_{R(A)}$ and $A^\dagger A = P_{R(A^T)}$. A matrix is called *semimonotone* if A has the non-negative Moore-Penrose inverse. We refer to [2] for more detail. Similarly, a square matrix A is called *monotone* if A^{-1} exists and $A^{-1} \geq 0$ (see [9]).

We next turn our attention to results related to proper splittings.The first one says if $A = U - V$ is a proper splitting of $A \in \mathbb{R}^{m\times n}$, then $A = U(I - U^\dagger V)$, $I - U^\dagger V$ is invertible and $A^\dagger = (I - U^\dagger V)^{-1}U^\dagger$. This is proved in [3], Theorem 1. Similarly, Climent and Perea [6] proved that $A = (I - VU^\dagger)U$ and $A^\dagger = U^\dagger(I - VU^\dagger)^{-1}$ for a proper splitting $A = U - V$.

For all proper splittings, the iteration scheme (1.2) may not converge. So, different convergence conditions are obtained for different subclasses of proper splittings by several authors starting with Berman and Plemmons [3]. We first collect below three such subclasses and then convergence criteria for the same subclasses.

Definition 2.1. A proper splitting $A = U - V$ of $A \in \mathbb{R}^{m\times n}$ is called
(i) a proper regular splitting if $U^\dagger \geq 0$ and $V \geq 0$. ([13])
(ii) a proper weak regular splitting of type I if $U^\dagger \geq 0$ and $U^\dagger V \geq 0$. ([6])
(iii) a proper weak regular splitting of type II if $U^\dagger \geq 0$ and $VU^\dagger \geq 0$. ([6])

Next one combines [3, Corollary 4] and [10, Theorem 3.7], and contains convergence criteria for both the above subclasses.

Theorem 2.2. *Let $A = U - V$ be a proper weak regular splitting of either type I or type II of $A \in \mathbb{R}^{m\times n}$. Then, A is semimonotone if and only if $\rho(U^\dagger V) < 1$.*

3. Main Results

This section have two parts. In the first part, we reprove a result by dropping one assumption and providing a complete new proof. We then present another comparison result. In the second part, we discuss theory of proper multisplittings.

3.1. **Comparison Results.** Comparison of the spectral radii of two proper splittings are useful for improving the speed of the iteration scheme (1.2). In this direction, several comparison results have been introduced in the literature both in rectangular and square nonsingular matrix setting. Very recently, Giri and Mishra [10] proved the following comparison result which extends [19, Theorem 3.7] to the rectangular case.

Theorem 3.1. [10, Theorem 3.13]
Let $A = U_1 - V_1 = U_2 - V_2$ be two proper weak regular splittings of different types of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$. Suppose that no row or column of $A^\dagger$ is zero. If $U_2^\dagger \leq U_1^\dagger$, then $\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1$.

We next provide an example where the condition "no row or column of $A^\dagger$ is zero" in Theorem 3.1 fails, but the conclusion holds.

Example 3.2. Let $A = \begin{pmatrix} 6 & -2 & 0 \\ -3 & 4 & 0 \end{pmatrix} = U_1 - V_1 = U_2 - V_2$, where $U_1 = \begin{pmatrix} 7 & -1 & 0 \\ -3 & 4 & 0 \end{pmatrix}$ and $U_2 = \begin{pmatrix} 14 & -2 & 0 \\ -9 & 12 & 0 \end{pmatrix}$, respectively. Then

$$R(U_1) = R(U_2) = R(A),\ N(U_1) = N(U_2) = N(A),\ U_1^\dagger = \begin{pmatrix} 0.1600 & 0.0400 \\ 0.1200 & 0.2800 \\ 0 & 0 \end{pmatrix} \geq 0,$$

$$U_1^\dagger V_1 = \begin{pmatrix} 0.1600 & 0.1600 & 0 \\ 0.1200 & 0.1200 & 0 \\ 0 & 0 & 0 \end{pmatrix} \geq 0,$$

$$U_2^\dagger = \begin{pmatrix} 0.0800 & 0.0133 \\ 0.0600 & 0.0933 \\ 0 & 0 \end{pmatrix} \geq 0,\ \ V_2U_2^\dagger = \begin{pmatrix} 0.6400 & 0.1067 \\ 0.0000 & 0.6667 \end{pmatrix} \geq 0.$$

Hence, $A = U_1 - V_1$ is a proper weak regular splitting of type I and $A = U_2 - V_2$ is a proper weak regular splitting of type II. Also $A^\dagger = \begin{pmatrix} 0.2222 & 0.1111 \\ 0.1667 & 0.3333 \\ 0 & 0 \end{pmatrix} \geq 0$ and $U_1^\dagger = \begin{pmatrix} 0.1600 & 0.0400 \\ 0.1200 & 0.2800 \\ 0 & 0 \end{pmatrix} \geq U_2^\dagger = \begin{pmatrix} 0.0800 & 0.0133 \\ 0.0600 & 0.0933 \\ 0 & 0 \end{pmatrix}$. But $0.2800 = \rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) = 0.6667 < 1$.

This leads to the fact that Theorem 3.1 may be true even without the assumption "no row or column of $A^\dagger$ is zero". This is stated and proved in the next result. The technique used in this proof is different from the earlier proof.

Theorem 3.3. *Let $A = U_1 - V_1 = U_2 - V_2$ be two proper weak regular splittings of different types of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$. If $U_2^\dagger \leq U_1^\dagger$, then*

$$\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1.$$

Proof. Let us first consider that $A = U_1 - V_1$ is a proper weak regular splitting of type I and $A = U_2 - V_2$ is a proper weak regular splitting of type II. We then have $\rho(U_1^\dagger V_1) < 1$ and $\rho(V_2U_2^\dagger) < 1$ by Theorem 2.2. The conditions $U_1^\dagger V_1 \geq 0$ and $\rho(U_1^\dagger V_1) < 1$ imply $(I - U_1^\dagger V_1)^{-1} \geq 0$. Similarly, $(I - V_2U_2^\dagger)^{-1} \geq 0$. Now, postmultiplying $U_2^\dagger \leq U_1^\dagger$ by $(I - V_2U_2^\dagger)^{-1}$, we obtain

$$A^\dagger = U_2^\dagger(I - V_2U_2^\dagger)^{-1} \leq U_1^\dagger(I - V_2U_2^\dagger)^{-1}, \tag{3.1}$$

and then premultiplying (3.1) by $(I - U_1^\dagger V_1)^{-1}$, we get

$$(I - U_1^\dagger V_1)^{-1}A^\dagger \leq (I - U_1^\dagger V_1)^{-1}U_1^\dagger(I - V_2U_2^\dagger)^{-1} = A^\dagger(I - V_2U_2^\dagger)^{-1}. \tag{3.2}$$

Since $U_1^\dagger V_1 \geq 0$, there exists an eigenvector $x \geq 0$ such that

$$x^TU_1^\dagger V_1 = \rho(U_1^\dagger V_1)x^T.$$

So, $x \in R(V_1^T) \subseteq R(A^T)$. Premultiplying (3.2) by x^T, we have

$$\frac{1}{1 - \rho(U_1^\dagger V_1)} x^TA^\dagger \leq x^TA^\dagger(I - V_2U_2^\dagger)^{-1}.$$

From [4, Theorem 2.1.11], we obtain

$$\frac{1}{1 - \rho(U_1^\dagger V_1)} \leq \frac{1}{1 - \rho(V_2U_2^\dagger)} = \frac{1}{1 - \rho(U_2^\dagger V_2)}, \tag{3.3}$$

as $x^TA^\dagger \geq 0$ and $x^TA^\dagger \neq 0$. Suppose that $x^TA^\dagger = 0$, then $x^TA^\dagger A = 0$, i.e., $(A^\dagger A)^Tx = A^\dagger Ax = x = 0$, a contradiction. Hence $x^TA^\dagger \neq 0$. Now, the desired result follows immediately from (3.3). The proof for the other types of splittings can be done similarly. □

The next example shows that the converse of the above result is not true.

Example 3.4. Let $A = \begin{pmatrix} 7 & -3 & 7 \\ -2 & 8 & -2 \end{pmatrix} = U_1 - V_1 = U_2 - V_2$, where $U_1 = \begin{pmatrix} 21 & -6 & 21 \\ -6 & 16 & -6 \end{pmatrix}$ and $U_2 = \begin{pmatrix} 14 & -6 & 14 \\ -8 & 32 & -8 \end{pmatrix}$, respectively. Then $R(U_1) = R(U_2) = R(A)$, $N(U_1) = N(U_2) = N(A)$, $U_1^\dagger = \begin{pmatrix} 0.0267 & 0.0100 \\ 0.0200 & 0.0700 \\ 0.0267 & 0.0100 \end{pmatrix} \geq 0$, $U_1^\dagger V_1 = \begin{pmatrix} 0.3333 & 0 & 0.3333 \\ 0 & 0.5000 & 0 \\ 0.3333 & 0 & 0.3333 \end{pmatrix} \geq 0$, $V_2 U_2^\dagger = \begin{pmatrix} 0.5000 & 0 \\ 0 & 0.7500 \end{pmatrix} \geq 0$. Hence, $A = U_1 - V_1$ is a proper weak regular splitting of type I and $A = U_2 - V_2$ is a proper weak regular splitting of type II. Also $A^\dagger = \begin{pmatrix} 0.0800 & 0.0300 \\ 0.0400 & 0.1400 \\ 0.0800 & 0.0300 \end{pmatrix} \geq 0$ and $\rho(U_1^\dagger V_1) = 0.6667 < \rho(U_2^\dagger V_2) = 0.7500 < 1$. But $U_2^\dagger = \begin{pmatrix} 0.0400 & 0.0075 \\ 0.0200 & 0.0350 \\ 0.0400 & 0.0075 \end{pmatrix} \not\leq U_1^\dagger = \begin{pmatrix} 0.0267 & 0.0100 \\ 0.0200 & 0.0700 \\ 0.0267 & 0.0100 \end{pmatrix}$.

For two proper weak regular splittings of the same type, we have the following comparison result.

Theorem 3.5. *Let $A = U_1 - V_1 = U_2 - V_2$ be two proper weak regular splittings of the same type of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$. If there is an α, $0 < \alpha \leq 1$ such that*

$$U_1 \leq \alpha U_2,$$

then

$$\rho(U_1^\dagger V_1) \leq \rho(U_2^\dagger V_2) < 1, \text{ whenever } \alpha = 1 \text{ and}$$
$$\rho(U_1^\dagger V_1) < \rho(U_2^\dagger V_2) < 1, \text{ whenever } 0 < \alpha < 1.$$

Proof. Assume that the given splittings are proper weak regular of type I and the condition $U_1 \leq \alpha U_2$ holds. Premultiplying $U_1 \leq \alpha U_2$ by $A^\dagger$, we obtain

$$A^\dagger U_1 \leq \alpha A^\dagger U_2, \text{ i.e.,}$$
$$(I - U_1^\dagger V_1)^{-1} U_1^\dagger U_1 \leq \alpha (I - U_2^\dagger V_2)^{-1} U_2^\dagger U_2. \tag{3.4}$$

Since $U_1^\dagger V_1 \geq 0$, there exists a non-negative eigenvector x such that $U_1^\dagger V_1 x = \rho(U_1^\dagger V_1)x$. Now, postmultiplying (3.4) by x, we obtain

$$\frac{x}{1 - \rho(U_1^\dagger V_1)} \leq \alpha (I - U_2^\dagger V_2)^{-1} x,$$

which implies

$$\frac{1}{1 - \rho(U_1^\dagger V_1)} \leq \frac{\alpha}{1 - \rho(U_2^\dagger V_2)},$$

by [4, Theorem 2.1.11]. Hence

$$(1-\alpha)+\alpha\rho(U_1^{\dagger}V_1)\leq\rho(U_2^{\dagger}V_2).$$

Now, the required result follows immediately. For the case, when the given splittings are proper weak regular with type II the proof is similar. □

Theorem 3.5 is also true if we replace the condition *the same type* by *different types*. Note that for the square nonsingular case, Song [17] proved a similar result(i.e., a part of Theorem 2.11) but for nonnegative splittings (see Definition 2.1 (iv), [17] for its definition).

3.2. **Proper Multisplitting Theory.** We next proceed to discuss proper multisplitting theory. The definition of a proper multisplitting of a rectangular matrix introduced by Climent and Perea [8] is as follows:

Definition 3.6. [8, Definition 2]
The triplet $(U_l,V_l,E_l)_{l=1}^{p}$ is a proper multisplitting of $A\in\mathbb{R}^{m\times n}$ if
(i) $A=U_l-V_l$ is a proper splitting, for each $l=1,2,\ldots,p$,
(ii) $E_l\geq 0$, for each $l=1,2,\ldots,p$ is a diagonal $n\times n$ matrix, and $\sum_{l=1}^{p}E_l=I$,
where I is the $n\times n$ identity matrix.

A proper multisplitting is called a *proper regular multisplitting* or *a proper weak regular multisplitting of type I*, if each one of the proper splitting $A=U_l-V_l$ is a proper regular splitting or a proper weak regular splitting of type I, respectively. Climent and Perea [8] considered the following parallel iterative scheme:

$$x^{k+1}=Hx^k+Gb,\ k=1,2,\ldots, \tag{3.5}$$

where $(U_l,V_l,E_l)_{l=1}^{p}$ is a proper multisplitting of $A\in\mathbb{R}^{m\times n}$, $H=\sum_{l=1}^{p}E_lU_l^{\dagger}V_l$ and $G=\sum_{l=1}^{p}E_lU_l^{\dagger}$. Now, we have the following convergence result for a proper multisplitting which generalizes a result stated in the introduction part of [7] to rectangular matrices.

Lemma 3.7. *Let $(U_l,V_l,E_l)_{l=1}^{p}$ be a proper multisplitting of $A\in\mathbb{R}^{m\times n}$. Then, the iterative scheme (3.5) converges to $A^{\dagger}b$ for every x^0 if and only if $\rho(H)<1$.*

Proof. We have $(I - U_l^\dagger V_l)A^\dagger = U_l^\dagger$ for each $l = 1, 2, \ldots, p$. So,

$$\begin{aligned} G &= \sum_{l=1}^{p} E_l U_l^\dagger \\ &= \sum_{l=1}^{p} E_l (I - U_l^\dagger V_l) A^\dagger \\ &= [\sum_{l=1}^{p} E_l - \sum_{l=1}^{p} E_l U_l^\dagger V_l] A^\dagger \\ &= (I - H)A^\dagger. \end{aligned}$$

Suppose that the iterative scheme (3.5) converges to $A^\dagger b$ for any initial vector x^0. To prove $\rho(H) < 1$, we show that for any $y \in \mathbb{R}^n$, $\lim_{k\to\infty} H^k y = 0$. Let $y \in \mathbb{R}^n$ be an arbitrary vector, and x be the unique least squares solution to (3.5). Define $x^0 = x - y$, and, for $k \geq 1$, $x^k = Hx^{k-1} + Gb$. Then (x^k) converges to x. Also,

$$x - x^k = (Hx + Gb) - (Hx^{k-1} + Gb) = H(x - x^{k-1}),$$

so

$$x - x^k = H(x - x^{k-1}) = H^2(x - x^{k-2}) = \cdots = H^k(x - x^0) = H^k y.$$

Hence $\lim_{k\to\infty} H^k y = \lim_{k\to\infty} H^k(x - x^0) = \lim_{k\to\infty} (x - x^k) = 0$. Hence $\rho(H) < 1$ by [5, Theorem 7.17].

Conversely, let $\rho(H) < 1$ and x^0 be any initial vector. From (3.5), we have

$$x^i = H^i x^0 + (I + H + \cdots + H^{i-1})Gb.$$

Since $\rho(H) < 1$, the matrix H is convergent, and $\lim_{i\to\infty} H^i x^0 = 0$ by [5, Theorem 7.17]. So $(I - H)^{-1} = \sum_{i=1}^{\infty} H^i$ by [4, Lemma 6.2.1]. Hence

$$\lim_{i\to\infty} x^i = \lim_{i\to\infty} H^i x^0 + \left(\sum_{i=0}^{\infty} H^i\right) Gb = (I - H)^{-1} Gb = A^\dagger b.$$

□

The next result is obtained as a corollary in the case of a nonsingular matrix A.

Corollary 3.8. ([7])
Let $(U_l, V_l, E_l)_{l=1}^p$ be a multisplitting of $A \in \mathbb{R}^{n\times n}$. Then, the iterative scheme (3.5) converges to $A^{-1}b$ for every x^0 if and only if $\rho(H) < 1$.

The next result presented below extends [16, Theorem 1 (a)] to rectangular matrices which is a characterization of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$.

Theorem 3.9. *Let $(U_l, V_l, E_l)_{l=1}^p$ be a proper weak regular multisplitting of type I of $A \in \mathbb{R}^{m\times n}$. Then, $A^\dagger \geq 0$ if and only if $\rho(H) < 1$.*

Proof. The first part is shown in [8, Theorem 4].

Conversely, since $(U_l, V_l, E_l)_{l=1}^p$ is a proper weak regular multisplitting of type I, we have $H \geq 0$ and $G \geq 0$. Assume that $\rho(H) < 1$. By [4, Lemma 6.2.1], $(I-H)^{-1} \geq 0$. Then $A^\dagger = (I-H)^{-1}G \geq 0$. □

For nonsingular case, we have the following corollary.

Corollary 3.10. [16, Theorem 1 (a)]
Let $(U_l, V_l, E_l)_{l=1}^p$ be a weak regular multisplitting of type I of $A \in \mathbb{R}^{n\times n}$. Then, $A^{-1} \geq 0$ if and only if $\rho(H) < 1$.

In the following result, we introduce an upper bound and a lower bound for the spectral radius of the iteration matrix H by extending [7, Theorem 3.4].

Theorem 3.11. *Let $(U_l, V_l, E_l)_{l=1}^p$ be a proper weak regular multisplitting of type I and $A = \overline{U} - \overline{V} = \underline{U} - \underline{V}$ be two proper weak regular splittings of type I of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$. If one of the following conditions holds.*

(i) $\overline{U}^\dagger A \leq U_l^\dagger A \leq \underline{U}^\dagger A,$ *for each* $l = 1, 2, \ldots, p.$

(ii) $(\overline{U}^\dagger A)^T \leq U_l^\dagger A \leq (\underline{U}^\dagger A)^T,$ *for each* $l = 1, 2, \ldots, p.$

Then $\rho(\underline{U}^\dagger \underline{V}) \leq \rho(H) \leq \rho(\overline{U}^\dagger \overline{V}) < 1.$

Proof. By Theorem 3.9, the given multisplitting is convergent, and by Theorem 2.2, the splittings $A = \overline{U} - \overline{V} = \underline{U} - \underline{V}$ are also convergent. For any proper splitting $A = U - V$, we obtain $U^\dagger A = (I - U^\dagger V)A^\dagger A$. So, $(I - \overline{U}^\dagger \overline{V})A^\dagger A \leq (I - U_l^\dagger V_l)A^\dagger A \leq (I - \underline{U}^\dagger \underline{V})A^\dagger A$ which implies $0 \leq \underline{U}^\dagger \underline{V} \leq U_l^\dagger V_l \leq \overline{U}^\dagger \overline{V}$. Premultiplying by $\sum_{l=1}^p E_l$, we get $0 \leq \underline{U}^\dagger \underline{V} \leq H \leq \overline{U}^\dagger \overline{V}$. Hence $\rho(\underline{U}^\dagger \underline{V}) \leq \rho(H) \leq \rho(\overline{U}^\dagger \overline{V}) < 1$ by [18, Theorem 2.21]. The proof of 2nd part follows similarly due to the fact that $\rho(B^T) = \rho(B)$. □

A result showing an upper bound and a lower bound for the spectral radius of H is illustrated below which improves the first part of [7, Theorem 3.3].

Theorem 3.12. *Let $(U_l, V_l, E_l)_{l=1}^p$ be a proper weak regular multisplitting of type I of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$ and $A = \overline{U} - \overline{V} = \underline{U} - \underline{V}$ be two proper weak regular splittings of type II. If $R(E_l) \subseteq R(A^T)$ and $\underline{U} \leq U_l \leq \overline{U}$, for each $l = 1, 2, \ldots, p$, then*

$$\rho(\underline{U}^\dagger \underline{V}) \leq \rho(H) \leq \rho(\overline{U}^\dagger \overline{V}) < 1.$$

Proof. By Theorem 2.2, it follows that $\rho(\overline{U}^\dagger \overline{V}) < 1$ and $\rho(\underline{U}^\dagger \underline{V}) < 1$. Premultiplying $U_l \leq \overline{U}$ by $U_l^\dagger$ and postmultiplying the same by $\overline{U}^\dagger$, we have $U_l^\dagger U_l \overline{U}^\dagger \leq U_l^\dagger \overline{U}\ \overline{U}^\dagger$, i.e., $\overline{U}^\dagger \leq U_l^\dagger$. Let $U_1^\dagger = \displaystyle\sum_{l=1}^p E_l U_l^\dagger$ and $U_2^\dagger = \overline{U}^\dagger$. Then, by [11, Theorem 3.21], it follows that $U_1^\dagger V_1 = H$. Since $U_l^\dagger \geq \overline{U}^\dagger$, we obtain $U_1^\dagger \geq U_2^\dagger$. Hence, by Theorem 3.3, we get $\rho(H) \leq \rho(\overline{U}^\dagger \overline{V})$. Similarly, by premultiplying $U_l^\dagger$ and postmultiplying $\underline{U}^\dagger$ to $\underline{U} \leq U_l$, we obtain $U_l^\dagger \underline{U}\ \underline{U}^\dagger \leq U_l^\dagger U_l \underline{U}^\dagger$, i.e., $U_l^\dagger \leq \underline{U}^\dagger$. Let

$U_1^\dagger = \underline{U}^\dagger$, $U_2^\dagger = \sum_{l=1}^{p} E_l U_l^\dagger$. Then $U_1^\dagger \geq U_2^\dagger$. So, by Theorem 3.3, we get $\rho(\underline{U}^\dagger \underline{V}) \leq \rho(H)$. Combining both the cases, we obtain $\rho(\underline{U}^\dagger \underline{V}) \leq \rho(H) \leq \rho(\overline{U}^\dagger \overline{V}) < 1$. □

Recently, Giri and Mishra [11] proved that the iteration matrix H in (3.5) induces a unique proper weak regular splitting of type I under some sufficient conditions. Next, we prove that the induced splitting in [11, Theorem 3.21], is also a proper regular splitting under the assumption of an extra sufficient condition.

Theorem 3.13. *Let $(U_l, V_l, E_l)_{l=1}^{p}$ be a proper weak regular multisplitting of type I of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$. If $R(E_l) \subseteq R(A^T)$ and there exists a non-negative matrix $M \in \mathbb{R}^{m\times n}$ such that $(A+M)U_l^\dagger V_l \geq M$, for each $l = 1, 2, \ldots, p$, then the unique splitting $A = B - C$ induced by H with $B = A(I - H)^{-1}$ is a proper regular splitting.*

Proof. By [11, Theorem 3.21], we have $\rho(H) < 1$ and the unique splitting $A = B - C$ induced by H is a proper weak regular of type I. Then

$$\begin{aligned} C &= B - A \\ &= A(I - H)^{-1} - A \\ &= AH(I - H)^{-1} \\ &\geq M(I - H)(I - H)^{-1} \\ &= M \geq 0 \end{aligned}$$

and thus $A = B - C$ is proper regular. □

The following example illustrates Theorem 3.13.

Example 3.14. Let $A = \begin{pmatrix} 2 & -3 \\ 0 & 5 \\ 4 & -6 \end{pmatrix}$. Set $U_1 = \begin{pmatrix} 4 & -3 \\ 0 & 5 \\ 8 & -6 \end{pmatrix}$, $U_2 = \begin{pmatrix} 6 & -3 \\ 0 & 5 \\ 12 & -6 \end{pmatrix}$, $E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $E_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, respectively. Then $R(U_1) = R(U_2) = R(A)$, $N(U_1) = N(U_2) = N(A)$, $U_1^\dagger = \begin{pmatrix} 0.0500 & 0.1500 & 0.1000 \\ 0 & 0.2000 & 0 \end{pmatrix} \geq 0$, $U_1^\dagger V_1 = \begin{pmatrix} 0.5000 & 0 \\ 0 & 0 \end{pmatrix} \geq 0$, $U_2^\dagger = \begin{pmatrix} 0.0333 & 0.1000 & 0.0667 \\ 0 & 0.2000 & 0 \end{pmatrix} \geq 0$ and $U_2^\dagger V_2 = \begin{pmatrix} 0.6667 & 0 \\ 0.0000 & 0 \end{pmatrix} \geq 0$. Hence, (U_k, V_k, E_k) is a proper weak regular multisplitting of type I with $R(E_k) \subseteq R(A^T)$ for each $k = 1, 2$. Let $M = \begin{pmatrix} 1.5 & 0 \\ 0 & 0 \\ 2.5 & 0 \end{pmatrix}$.

Then $(A+M)U_1^{\dagger}V_1 = \begin{pmatrix} 1.75 & 0 \\ 0 & 0 \\ 2.5 & 0 \end{pmatrix} \geq \begin{pmatrix} 1.5 & 0 \\ 0 & 0 \\ 2.5 & 0 \end{pmatrix} = M$ and $(A+M)U_2^{\dagger}V_2 = \begin{pmatrix} 2.3333 & 0 \\ 0.0000 & 0 \\ 3.3333 & 0 \end{pmatrix} \geq \begin{pmatrix} 1.5 & 0 \\ 0 & 0 \\ 2.5 & 0 \end{pmatrix} = M$. So, it satisfies all the conditions of Theorem 3.13. Therefore, the unique induced splitting $A = B - C$ is proper regular, as $R(B) = R(A)$, $N(B) = N(A)$, $B^{\dagger} = \begin{pmatrix} 0.05 & 0.1500 & 0.10 \\ 0 & 0.2 & 0 \end{pmatrix} \geq 0$ and $C = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 4 & 0 \end{pmatrix} \geq 0$.

Theorem 3.13 admits the following corollary in the case of nonsingular matrices.

Corollary 3.15. *Let $(U_l, V_l, E_l)_{l=1}^{p}$ be a weak regular multisplitting of type I of a monotone matrix $A \in \mathbb{R}^{n\times n}$. If there exists a non-negative matrix $M \in \mathbb{R}^{n\times n}$ such that $(A+M)U_l^{-1}V_l \geq M$, for each $l = 1, 2, \ldots, p$, then the unique splitting $A = B - C$ induced by H with $B = A(I-H)^{-1}$ is a regular splitting.*

By substituting $M = 0$ in Theorem 3.13, we have the following result.

Corollary 3.16. *Let $(U_l, V_l, E_l)_{l=1}^{p}$ be a proper weak regular multisplitting of type I of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$. If $R(E_l) \subseteq R(A^T)$ and $V_l U_l^{\dagger} V_l \leq V_l$, for each $l = 1, 2, \ldots, p$, then the unique splitting $A = B - C$ induced by H with $B = A(I-H)^{-1}$ is a proper regular splitting.*

Theorem 3.13 is again proved below under the assumption of different conditions by generalizing [14, Theorem 3.3].

Theorem 3.17. *Let $(U_l, V_l, E_l)_{l=1}^{p}$ be a proper weak regular multisplitting of type I of a semimonotone matrix $A \in \mathbb{R}^{m\times n}$. Assume that, for each l, $E_l = \alpha_l I$ with $\alpha_l > 0$ and $\sum_{l=1}^{p} \alpha_l = 1$. Let $V_a = \sum_{l=1}^{p} \alpha_l V_l$, and $V_b \leq V_l$, for each $l = 1, 2, \ldots, p$. If there exists a non-negative matrix $U \in \mathbb{R}^{m\times n}$ such that $(V_b - U)U_l^{\dagger} V_l \leq V_a - U$, for each $l = 1, 2, \ldots, p$, then the unique splitting $A = B - C$ induced by H with $B = A(I-H)^{-1}$ is a proper regular splitting.*

Proof. As $C = B - A = AH(I-H)^{-1}$, we have

$$\begin{aligned} C &= \sum_{l=1}^{p} \alpha_l A U_l^{\dagger} V_l (I-H)^{-1} \\ &= \sum_{l=1}^{p} \alpha_l V_l (I - U_l^{\dagger} V_l)(I-H)^{-1} \\ &= \sum_{l=1}^{p} \alpha_l V_l (I-H)^{-1} - \sum_{l=1}^{p} \alpha_l V_l U_l^{\dagger} V_l (I-H)^{-1} \\ &\geq V_a (I-H)^{-1} - V_b \sum_{l=1}^{p} \alpha_l U_l^{\dagger} V_l (I-H)^{-1} \\ &= V_a (I-H)^{-1} - V_b H (I-H)^{-1} \\ &= (V_a - V_b H)(I-H)^{-1} \\ &\geq U \geq 0. \end{aligned}$$

□

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