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# LOCAL CONVERGENCE FOR A FAMILY OF SIXTH ORDER CHEBYSHEV-HALLEY -TYPE METHODS IN BANACH SPACE UNDER WEAK CONDITIONS

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Abstract. We present a local convergence analysis for a family of super-Halley methods of high convergence order in order to approximate a solution of a nonlinear equation in a Banach space. Our sufficient convergence conditions involve only hypotheses on the first and second Fréchet-derivative of the operator involved. Earlier studies use hypotheses up to the third Fréchet derivative. Numerical examples are also provided in this study.

## 1. Introduction

Many problems in computational sciences and other disciplines can be brought in a form of equation

$$F(x) = 0, \tag{1.1}$$

where $F$ is a Fréchet-differentiable operator defined on a subset $D$ of a Banach space $X$ with values in a Banach space $Y$. The solutions of these equation (1.1) can rarely be found in closed form. Therefore solutions of these equations (1.1) are approximated by Newton-like iterative methods [1–28]. The study about convergence matter of iterative procedures is usually based on two types: semi-local and local convergence analysis. The semi-local convergence matter is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information

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around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal with the local and semilocal convergence analysis of Newton-like methods such as [1–28].

We present a local convergence analysis for the family of Chebyshev-Halley-type methods defined for each $n = 0, 1, 2, \cdots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ u_n &= x_n - \frac{\beta}{2}F'(x_n)^{-1}F(x_n), \\ z_n &= y_n - A_nF'(x_n)^{-1}F(x_n), \\ x_{n+1} &= z_n - B_n^{-1}F'(x_n)^{-1}F(z_n), \end{aligned} \tag{1.2}$$

where $x_0$ is an initial point, $\alpha$, $\beta \in (-\infty, \infty)$,

$$A_n = \frac{1}{2}G(x_n)\triangle_{\alpha,n}^{-1},\ \triangle_{\alpha,n} = I - \alpha G(x_n),$$

$$G(x_n) = F'(x_n)^{-1}F''(u_n)F'(x_n)^{-1}F(x_n),$$

and $B_n = I + F'(x_n)^{-1}F''(u_n)(z_n - x_n)$. A semilocal convergence analysis was presented in [28] for the special case when $\beta = 1$ under the following conditions $(\mathcal{C}')$( in non affine invariant form): Let $F : D \subseteq X \to Y$ be a thrice differentiable operator.

**$(\mathcal{C}_1')$:** There exists $F'(x_0)^{-1} \in L(Y, X)$ and $\|F'(x_0)^{-1}\| \leq \beta$;

**$(\mathcal{C}_2')$:**

$$\|F'(x_0)^{-1}F(x_0)\| \leq \beta_1;$$

**$(\mathcal{C}_3')$:**

$$\|F'(x_0)^{-1}F''(x)\| \leq \beta_2 \quad \text{for each } \ x \in D;$$

**$(\mathcal{C}_4')$:**

$$\|F'(x_0)^{-1}F'''(x)\| \leq \beta_3 \quad \text{for each } \ x \in D;$$

**$(\mathcal{C}_5')$:**

$$\|F'(x_0)^{-1}(F'''(x) - F'''(y))\| \leq \beta_4\|x - y\|^q \quad \text{for each } \ x, y \in D \text{ and } q \in [0, 1].$$

The $R-$order of convergence was shown to be $5 + q$ (i.e., 6 if $q = 1$). Notice that in [28] $\alpha \in [0, 1]$, whereas in the present paper $\alpha \in (-\infty, \infty)$. Hence, the applicability of method (1.2) is extended.

It is worth noting that if $F''(x) = Q$ where $Q$ is a bilinear operator, then method (1.2) finds applications especially, when $F$ is a quadratic operator [2].

Similar conditions have been used by other authors [1–28], on other high convergence order methods. The corresponding conditions for the local convergence analysis are given by simply replacing $x_0$ by $x^*$ in the preceding $(\mathcal{C}')$ conditions. These conditions however are very restrictive. As a motivational example, let us define function $f$ on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} f'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \; f'(1) = 3, \\ f''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\ f'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22. \end{aligned}$$

Then, obviously, e.g, function $f$ cannot satisfy condition $(\mathcal{C}_4')$, say for $q = 1$, since function $f'''$ is unbounded on $D$. In the present paper we only use hypotheses on the first Fréchet derivative. In particular, we weaken the $(\mathcal{C}')$ hypotheses. We suppose instead (in affine invariant form) the conditions $(\mathcal{C})$: Let $F : D \subseteq X \to Y$ be a twice differentiable operator.

**($\mathcal{C}_1$):** There exists $x^* \in D$ such that $F'(x^*) = 0$ and $F'(x^*)^{-1} \in L(Y, X)$.
There exist $L_0 > 0$, $L > 0$, $M > 0$ and $N > 0$ such that for each $x \in D$

**($\mathcal{C}_2$):**

$$\|F'(x^*)^{-1}(F(x) - F(x^*) - F'(x))(x - x^*)\| \leq \frac{L}{2}\|x - x^*\|^2;$$

**($\mathcal{C}_3$):**

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0\|x - x^*\|;$$

**($\mathcal{C}_4$):**

$$\|F'(x^*)^{-1}F'(x)\| \leq M;$$

and

**($\mathcal{C}_5$):**

$$\|F'(x^*)^{-1}F''(x)\| \leq N.$$

This way we expand the applicability of method (1.2). It is worth noticing that the sixth order of convergence was obtained under the $(\mathcal{C}')$ conditions. Here, under the $(\mathcal{C})$ conditions and since we are using the same method (1.2) (at least for $\beta = 1$), we use instead the computational order of convergence or the approximate computational order of convergence to arrive at the convergence order six (see Remark 2.2). Another advantage of our approach is that our results are obtained in affine invariant form. The advantages of obtaining results in affine invariant form over non affine invariant form are well known in the literature [3, 16].

The rest of the paper is organized as follows. The local convergence of method (1.2) is given in Section 2, whereas the numerical examples are given in the concluding Section 3.

## 2. Local convergence

We present the local convergence analysis of method (1.2) in this section.

Let $\alpha, \beta \in (-\infty, \infty)$, $L_0 > 0$, $L > 0, M > 0$ and $N > 0$ be parameters with $L_0 \leq L$. It is convenient for the local convergence analysis that follows to introduce some functions $g_1$ and $g_2$ on the interval $[0, \frac{1}{L_0})$ by

$$g_1(r) = \frac{Lr}{2(1 - L_0 r)},$$

and

$$\begin{aligned} g_2(r) &= g_1(r) + \frac{|2-\beta|M}{2(1-L_0 r)} \\ &= \frac{Lr + M|2-\beta|}{2(1-L_0 r)}. \end{aligned}$$

It follows from the definition of function $g_1$ that for

$$r_A := \frac{2}{L + 2L_0} \tag{2.1}$$

we have

$$0 \le g_1(r) < 1 \text{ for each } r \in [0, r_A).$$

Notice that

$$r_R := \frac{2}{3L} \le \frac{2}{L+2L_0} < \frac{1}{L_0}.$$

Set

$$r_2 = \frac{2 - |2-\beta|M}{2L_0 + L}. \tag{2.2}$$

Suppose that

$$|2-\beta|M < 2. \tag{2.3}$$

Then, we have that

$$0 \le g_2(r) < 1 \text{ for each } r \in [0, r_2).$$

We also have that

$$r_2 \le r_A.$$

Define quadratic polynomial $p_3$ by

$$p_3(r) = |\alpha| M N r - (1 - L_0 r)^2.$$

We have that

$$p_3(0) = -1 < 0 \text{ and } p_3(\frac{1}{L_0}) = \frac{|\alpha| M N}{L_0} > 0,$$

for $\alpha \neq 0$. It follows from the intermediate value theorem that polynomial $p_3$ has roots in the interval $(0, \frac{1}{L_0})$. Denote by $r_3$ the smallest such root. Define function $g_3$ on the interval $[0, \frac{1}{L_0})$ by

$$g_3(r) = \frac{|\alpha| M N r}{(1 - L_0 r)^2}.$$

Then it follows from the definition of function $g_3$ and polynomial $p_3$ that

$$0 \le g_3(r) < 1 \text{ for each } r \in [0, r_3).$$

Define function $g_4$ on $[0, r_3)$ by

$$g_4(r) = \frac{MNr}{2[(1 - L_0 r)^2 - |\alpha| M N r]}.$$

Then, we have that

$$g_4(r) \ge 0 \text{ for each } r \in [0, r_3).$$

Similarly, we show that function $h_5(r) = g_5(r) - 1$ defined by

$$g_5(r) = g_1(r) + \frac{g_4(r)Mr}{1 - L_0 r} = \frac{(L + 2g_4(r)M)r}{2(1 - L_0 r)}$$

has a minimal zero $r_5$ such that

$$0 \leq g_5(r) < 1 \text{ for each } r \in [0, r_5).$$

Indeed, we have that $h_5(0) = -1 < 0$ and $h_5(t) \to \infty$ as $t \to (\frac{1}{L_0})^-$. Similarly, functions $h_6(r) = g_6(r) - 1$, $h_7(r) = g_7(r) - 1$ and $h_8(r) = g_8(r) - 1$ have minimal zeros $r_6, r_7$ and $r_8$ such that

$$0 \leq g_6(r) < 1 \text{ for each } r \in [0, r_6),$$

$$0 \leq g_7(r) < 1 \text{ for each } r \in [0, r_7),$$

and

$$0 \leq g_8(r) < 1 \text{ for each } r \in [0, r_8),$$

where

$$g_6(r) = [(N + L_0) + N g_5(r)]r,$$

$$g_7(r) = \left(1 + \frac{M}{1 - [(N + L_0) + N g_5(r)]r}\right) g_5(r),$$

and

$$g_8(r) = 1 - [N + L_0 + N g_5(r)]r,$$

since we have that $h_6(0) = -1 < 0$, $h_6(t) \to \infty$, $h_7(0) = -1 < 0$, $h_7(t) \to \infty$, $h_8(0) = -1 < 0$ and $h_8(t) \to \infty$ as $t \to (\frac{1}{L_0})^-$. It follows from the definition of functions $g_1, g_5, g_7, g_8$ and parameters $r_A, r_5, r_7$ and $r_8$ that

$$r_7 \leq r_5 \leq r_A.$$

Set

$$r^* = \min\{r_2, r_3, r_6, r_7, r_8\}. \tag{2.4}$$

Then it follows from (2.3) and (2.4) that

$$0 \leq \quad g_1(r) \quad < 1, \tag{2.5}$$

$$0 \leq \quad g_2(r) \quad < 1, \tag{2.6}$$

$$0 \leq \quad g_3(r) \quad < 1, \tag{2.7}$$

$$g_8(r) > 0, \quad g_4(r) \quad > 0, \tag{2.8}$$

$$0 \leq \quad g_5(r) \quad < 1, \tag{2.9}$$

$$0 \leq \quad g_6(r) \quad < 1, \tag{2.10}$$

and

$$0 \leq g_7(r) < 1, \tag{2.11}$$

for each $r \in [0, r^*)$.

We shall denote by $U(q, R)$, $\bar{U}(q, R)$ the open and closed balls in $X$, respectively, with center $q \in X$ and of radius $R > 0$.

Next, we present the local convergence analysis of method (1.2) under the $(\mathcal{C})$ conditions.

**Theorem 2.1.** *Let $F : D \subseteq X \to Y$ be a twice Fréchet-differentiable operator. Suppose that the ($\mathcal{C}$) conditions, $|2-\beta|M < 2$ and*

$$\bar{U}(x^*, r^*) \subseteq D \tag{2.12}$$

*hold, where $r^*$ is defined by (2.4). Then, sequence $\{x_n\}$ generated by method (1.2) for $x_0 \in U(x^*, r^*)$ is well defined, remains in $U(x^*, r^*)$ and converges to $x^*$. Moreover, the following estimates hold for each $n = 0, 1, 2, \cdots$,*

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r^*, \tag{2.13}$$

$$\|u_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.14}$$

$$|\alpha|\|G(x_n)\| \leq g_3(\|x_n - x^*\|) < 1, \tag{2.15}$$

$$\|A_n\| \leq g_4(\|x_n - x^*\|), \tag{2.16}$$

$$\|z_n - x^*\| \leq g_5(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.17}$$

$$0 < g_6(\|x_n - x^*\|) < 1 \tag{2.18}$$

*and*

$$\|x_{n+1} - x^*\| \leq g_7(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.19}$$

*where the "g" functions are defined above Theorem 2.1. Furthermore, suppose that there exists $R \in [r^*, \frac{2}{L_0})$ such that $\bar{U}(x^*, R) \subset D$, then the limit point $x^*$ is the only solution of equation $F(x) = 0$ in $\bar{U}(x^*, R)$.*

*Proof.* Using ($\mathcal{C}_3$), the definition of $r^*$ and the hypothesis $x_0 \in U(x^*, r^*)$ we get that

$$\|F'(x^*)^{-1}(F(x_0) - F(x^*))\| \leq L_0\|x_0 - x^*\| < L_0 r^* < 1. \tag{2.20}$$

It follows from (2.20) and the Banach Lemma on invertible operators [3, 16] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0\|x_0 - x^*\|} < \frac{1}{1 - L_0 r^*}. \tag{2.21}$$

Hence, from the first two substeps of method (1.2) for $n = 0, y_0$ and $u_0$ are well defined. Using the first substep of method (1.2) for $n = 0$, and ($\mathcal{C}_1$) we first get the identity

$$\begin{aligned} & y_0 - x^* \\ = \; & x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ = \; & -F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^*)]. \end{aligned} \tag{2.22}$$

Then, by ($\mathcal{C}_2$), (2.21), (2.22), the definition of $r^*$ and (2.5) we get in turn that

$$\begin{aligned} \|y_0 - x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^*)]\| \\ &\leq \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r^*, \end{aligned}$$

which shows (2.13) for $n=0$. Using the second substep of method (1.2) for $n=0$, (2.13), $(\mathcal{C}_4)$, (2.21) and (2.6), we get that

$$\begin{aligned}\|u_0-x^*\| &\le \|y_0-x^*\|+\frac{|2-\alpha|}{2}\|F'(x_0)^{-1}F'(x^*)\|\\ &\quad\times\|F'(x^*)^{-1}F'(x_0)\|\\ &\le g_1(\|x_0-x^*\|)\|x_0-x^*\|+\frac{|2-\beta|}{2(1-L_0\|x_0-x^*\|)}\\ &\quad\times\|\int_0^1 F'(x^*)^{-1}F'(x^*+t(x_0-x^*))dt\|\|x_0-x^*\|\\ &= (g_1(\|x_0-x^*\|)+\frac{|2-\beta|M}{2(1-L_0\|x_0-x^*\|)})\|x_0-x^*\|\\ &\le g_2(\|x_0-x^*\|)\|x_0-x^*\|<\|x_0-x^*\|<r^*,\end{aligned}$$

which shows (2.14) for $n=0$. Notice also that $x^*+t(x_0-x^*)\in U(x^*,r^*)$, since

$$\|x^*+t(x_0-x^*)-x^*\|=|t|\|x_0-x^*\|<r^*$$

for each $t\in[0,1]$. We also have shown that $y_0,u_0\in U(x^*,r^*)$. We need estimates on $\|G(x_0)\|$, $\|A_0\|$, $\|\triangle_{\alpha,0}^{-1}\|$ and $\|B_0^{-1}\|$. Using $(\mathcal{C}_5)$, (2.21) and (2.7), we get that

$$\begin{aligned}|\alpha|\|G(x_0)\| &\le |\alpha|\|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F''(u_0)\|\\ &\quad \|F'(x_0)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F'(x_0)\|\\ &\le \frac{|\alpha|MN\|x_0-x^*\|}{(1-L_0\|x_0-x^*\|)^2}=g_3(\|x_0-x^*\|)<1,\end{aligned}\tag{2.23}$$

which shows (2.15) for $n=0$. It follows from (2.23) and the Banach lemma on invertible operators that $\triangle_{\alpha,0}^{-1}$ exists and

$$\|\triangle_{\alpha,0}^{-1}\|\le\frac{1}{1-\frac{|\alpha|MN\|x_0-x^*\|}{(1-L_0\|x_0-x^*\|)^2}}.\tag{2.24}$$

Then, using (2.23) and (2.26), we obtain that

$$\begin{aligned}\|A_0\| &\le \frac{MN\|x_0-x^*\|}{2[(1-L_0\|x_0-x^*\|)^2-|\alpha|MN\|x_0-x^*\|]}\\ &= g_4(\|x_0-x^*\|),\end{aligned}$$

which shows (2.16) for $n=0$. Then, using the third substep in method (1.2) for $n=0$, we see that $z_0$ is well defined. Moreover, from (2.13), (2.21), (2.9) and (2.16), we get that

$$\begin{aligned}\|z_0-x^*\| &\le \|x_0-F'(x_0)^{-1}F'(x^*)-x^*\|\\ &\quad+\|A_0\|\|F'(x^*)^{-1}F'(x^*)\|\|F'(x^*)^{-1}F'(x_0)\|\\ &\le (g_1(\|x_0-x^*\|)+\frac{g_4(\|x_0-x^*\|)M}{1-L_0\|x_0-x^*\|})\|x_0-x^*\|\\ &= g_5(\|x_0-x^*\|)\|x_0-x^*\|<\|x_0-x^*\|<r^*,\end{aligned}$$

which shows (2.17) for $n=0$. We have by (2.10) and (2.17) that

$$\begin{aligned} N\|z_0-x_0\|+L_0\|x_0-x^*\| &\leq N(\|z_0-x^*\|+\|x_0-x^*\|)+L_0\|x_0-x^*\| \\ &\leq (N+L_0)\|x_0-x^*\|+Ng_5(\|x_0-x^*\|)\|x_0-x^*\| \\ &= g_6(\|x_0-x^*\|)<1, \end{aligned} \tag{2.25}$$

which shows (2.18) for $n=0$. We also have from (2.25) and the Banach lemma on invertible operators that $B_0^{-1}$ exists and

$$\|B_0^{-1}\| \leq \frac{1}{1-(L_0\|x_0-x^*\|+N\|x_0-x^*\|)} \leq \frac{1}{1-g_6(\|x_0-x^*\|)}, \tag{2.26}$$

since by the definition of $B_0$

$$\begin{aligned} \|F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}F''(u_0)(z_0-x_0)\| &\leq \|F'(x_0)^{-1}F'(x^*)\| \\ &\quad \times\|F'(x^*)^{-1}F''(u_0)\|\|z_0-x_0\| \\ &\leq \frac{N\|z_0-x_0\|}{1-L_0\|x_0-x^*\|} \\ &\leq \frac{N(\|z_0-x^*\|+\|x_0-x^*\|)}{1-L_0\|x_0-x^*\|} \\ &< 1 \text{ (by (2.25))}. \end{aligned}$$

Then, using the last substep of method (1.2) for $n=0$, (2.11), (2.21) and (2.26), we get since $x_1$ is well defined that

$$\|x_1-x^*\| \leq g_5(\|x_0-x^*\|)\|x_0-x^*\|$$

$$+\|B_0^{-1}\|\|F'(x_0)^{-1}F'(x^*)\|\|\int_0^1 F'(x^*)^{-1}F'(x^*+t(z_0-x^*))(z_0-x^*)dt\|$$

$$\begin{aligned} &\leq g_5(\|x_0-x^*\|)\|x_0-x^*\| \\ &\quad +\frac{M\|z_0-x^*\|}{(1-L_0\|x_0-x^*\|)[1-\frac{N\|z_0-x_0\|}{1-L_0\|x_0-x^*\|}]} \\ &\leq (1+\frac{M}{1-(L_0\|x_0-x^*\|+N\|z_0-x^*\|)})g_5(\|x_0-x^*\|)\|x_0-x^*\| \\ &\leq g_7(\|x_0-x^*\|)\|x_0-x^*\| \\ &< \|x_0-x^*\|<r^*, \end{aligned}$$

which shows (2.19) for $n=0$.

By simply replacing $y_0, u_0, z_0, x_1$ by $y_k, u_k, z_k, x_{k+1}$ in the preceding estimates we arrive at (2.13)-(2.19). Moreover, from the estimate $\|x_{k+1}-x^*\|<\|x_k-x^*\|$, we deduce that $\lim_{k\to\infty} x_k = x^*$.

Finally, to show the uniqueness part, let $T = \int_0^1 F'(y^* + t(x^* - y^*))dt$ for some $y^* \in \bar{U}(x^*, R)$ with $F(y^*) = 0$. In view of $(\mathcal{C}_3)$ and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}(T - F'(x^*))\| &\leq \int_0^1 L_0\|y^* + t(x^* - y^*) - x^*\|dt \\ &\leq \int_0^1 L_0(1-t)\|x^* - y^*\|dt \leq \frac{L_0}{2}R < 1, \end{aligned}$$

it follows that $T^{-1}$ exists. Then, from the identity $0 = F(x^*) - F(y^*) = T(x^* - y^*)$, we deduce that $x^* = y^*$. □

*Remark* 2.2.

1. In view of $(\mathcal{C}_3)$ and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + L_0\|x - x^*\| \end{aligned}$$

   condition $(\mathcal{C}_4)$ can be dropped and $M$ can be replaced by

$$M(r) = 1 + L_0 r.$$

   Moreover, condition $(\mathcal{C}_2)$ can be replaced by the popular but stronger conditions

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L\|x - y\|, \quad \text{for each } x, y \in D \tag{2.27}$$

   or

$$\|F'(x^*)^{-1}(F'(x^* + t(x - x^*)) - F'(x))\| \leq L(1-t)\|x - x^*\|,$$

   for each $x, y \in D$ and $t \in [0, 1]$.
2. The results obtained here can be used for operators $F$ satisfying autonomous differential equations [3, 16] of the form

$$F'(x) = T(F(x))$$

   where $T$ is a continuous operator. Then, since $F'(x^*) = T(F(x^*)) = T(0)$, we can apply the results without actually knowing $x^*$. For example, let $F(x) = e^x - 1$. Then, we can choose: $T(x) = x + 1$.
3. The local results obtained here can be used for projection methods such as the Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method(GCR) for combined Newton/finite projection methods and in connection to the mesh independence principle can be used to develop the cheapest and most efficient mesh refinement strategies [3, 4].
4. The parameter $r_A$ given by (2.1) was shown by us to be the convergence radius of Newton's method [3, 4]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \ \text{ for each } \ n = 0, 1, 2, \cdots \tag{2.28}$$

   under the conditions (2.27) and $(\mathcal{C}_3)$. It follows from (2.4) that the convergence radius $r^*$ of method (1.2) cannot be larger than the convergence radius $r_A$ of the second order Newton's method (2.28). As already noted in [3, 4] $r_A$ is at least as large as the convergence ball given by Rheinboldt [3, 4]

$$r_R = \frac{2}{3L}.$$

In particular, for $L_0 < L$ we have that

$$r_R < r_A$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \quad as \quad \frac{L_0}{L} \to 0.$$

That is our convergence ball $r_A$ is at most three times larger than Rheinboldt's. The same value for $r_R$ was given by Traub [3, 4].

5. It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of the stronger $(\mathcal{C}')$ conditions used in [28]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) / \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) / \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds given in [28] involving estimates up to the third Fréchet derivative of operator $F$.

## 3. Numerical Examples

We present two numerical examples in this section.

**Example 3.1.** Let $X = Y = \mathbb{R}^2, D = \bar{U}(0,1)$, $x^* = 0$ and define function $F$ on $D$ for $z = (x, y)^T$ by

$$F(z) = (\sin x, \frac{1}{4}(e^y + 3y - 1))^T. \tag{3.1}$$

Then, using the $(\mathcal{C})$ conditions, we get $L_0 = L = 1$, $M = \frac{1}{4}(e+3)$, $N = \frac{e}{4}$. Then, we have $r_2 = 1.000$, $r_3 = 0.5048$, $r_5 = 0.3267$, $r_6 = 0.3325$, $r_7 = 0.1261$, $r_8 = 0.3212$, $\alpha = 0.5$, $\beta = 2.6995$,

$$r^* = 0.1261 < r_R = r_A = 0.6667$$

and

$$\xi_1 = 4.9590.$$

**Example 3.2.** Returning back to the motivational example at the introduction of this study, we have $L_0 = L = N = 146.6629073$, $M = 101.5578008$. Then we have $r_2 = 0.0068$, $r_3 = 0.0008$, $r_5 = 0.0005$, $r_6 = 0.0007$, $r_7 = 0.0236$, $r_8 = 0.0010$, $\alpha = 0.1$, $\beta = 2.0098$,

$$r^* = 0.0005 < r_R = r_A = 0.0045$$

and

$$\xi_1 = 3.8657.$$

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