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# STRONG DIFFERENTIAL SUBORDINATIONS FOR HIGHER-ORDER DERIVATIVES OF MULTIVALENT ANALYTIC FUNCTIONS DEFINED BY LINEAR OPERATOR

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Abstract. In the present paper, we introduce and study a new class of higher-order derivatives multivalent analytic functions in the open unit disk and closed unit disk of the complex plane by using linear operator. Also we obtain some interesting properties of this class and discuss several strong differential subordinations for higher-order derivatives of multivalent analytic functions.

## 1. Introduction and preliminaries

Let $U=\{z\in C:|z|<1\}$ and $\overline{U}=\{z\in C:|z|\leq 1\}$ denote the open unit disk and the closed unit disk of the complex plane, respectively. Denote by $\mathcal{H}\left(U\times\overline{U}\right)$ the class of analytic functions in $U\times\overline{U}$.

For $n$ a positive integer and $a\in C$, let

$$\mathcal{H}^*[a,n,\zeta]=\left\{f\in\mathcal{H}\left(U\times\overline{U}\right):f(z,\zeta)=a+a_n(\zeta)\,z^n+a_{n+1}(\zeta)\,z^{n+1}+\cdots,\right.$$
$$\left.z\in U,\zeta\in\overline{U}\right\},$$

where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k\geq n$.

Let $\mathcal{A}^*_{n\zeta}=\left\{f\in\mathcal{H}\left(U\times\overline{U}\right):f(z,\zeta)=a_{n+1}(\zeta)\,z^{n+1}+\cdots,z\in U,\zeta\in\overline{U}\right\}$, with $\mathcal{A}^*_{1\zeta}=\mathcal{A}^*_{\zeta}$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k\geq n+1$.

Denote by

$$S^*_\zeta=\left\{f\in\mathcal{H}^*[a,n,\zeta]:Re\left\{\frac{zf'_z(z,\zeta)}{f(z,\zeta)}\right\}>0,z\in U\ forall\ \zeta\in\overline{U}\right\}$$

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the class of starlike functions in $U \times \overline{U}$ and by

$$K_{\zeta}^{*}=\left\{f\in\mathcal{H}^{*}[a,n,\zeta]:Re\left\{\frac{zf''_{z^2}(z,\zeta)}{f'_{z}(z,\zeta)}+1\right\}>0,z\in U\ forall\ \zeta\in\overline{U}\right\}$$

the class of convex functions in $U \times \overline{U}$.

**Definition 1.1.** [8] Let $f(z,\zeta)$, $g(z,\zeta)$ be analytic in $U\times\overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $g(z,\zeta)$, written $f(z,\zeta)\prec\prec g(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, if there exists an analytic function $w$ in $U$ with $w(0)=0$ and $|w(z)|<1$, $z\in U$ such that $f(z,\zeta)=g(w(z),\zeta)$ for all $\zeta\in\overline{U}$.

*Remark* 1.2. [8]

(1) Since $f(z,\zeta)$ is analytic in $U\times\overline{U}$, for all $\zeta\in\overline{U}$ and univalent in $U$, for all $\zeta\in\overline{U}$, Definition 1.1 is equivalent to $f(0,\zeta)=g(0,\zeta)$ for all $\zeta\in\overline{U}$ and $f\left(U\times\overline{U}\right)\subset g\left(U\times\overline{U}\right)$.
(2) If $f(z,\zeta)=f(z)$ and $g(z,\zeta)=g(z)$, then the strong subordination becomes the usual notion of subordination.

Let $\mathcal{A}_{\zeta}^{*}(p)$ denote the subclass of the functions $f(z,\zeta)\in\mathcal{H}\left(U\times\overline{U}\right)$ of the form:

$$f(z,\zeta)=z^{p}+\sum_{k=1}^{\infty}a_{p+k}(\zeta)z^{p+k},\quad\left(p\in N=\{1,2,\cdots\},z\in U,\zeta\in\overline{U}\right),\qquad(1.1)$$

which are analytic and multivalent in $U\times\overline{U}$.
Upon differentiating both sides of (1.1) $j$-times with respect to $z$, we obtain

$$(f(z,\zeta))_{z^j}^{j}=\delta(p,j)z^{p-j}+\sum_{k=1}^{\infty}\delta(p+k,j)a_{p+k}(\zeta)z^{p+k-j},$$

$$(p\in N,j\in N_0=N\cup\{0\},p>j),$$

where

$$\delta(p,j)=\frac{p!}{(p-j)!}=\begin{cases}1 & (j=0)\\ p(p-1)\cdots(p-j+1), & (j\neq 0).\end{cases}$$

For $a\in R$, $c\in R\setminus Z_0^-$, where $Z_0^-=\{0,-1,-2,\cdots\}$, $0\le\lambda<1$, $p\in N$, $\alpha>-p$, $\mu,\nu\in R$ with $\mu-\nu-p<1$ and $\mathcal{A}_{\zeta}^{*}(p)$, the linear operator $\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c):\mathcal{A}_{\zeta}^{*}(p)\longrightarrow\mathcal{A}_{\zeta}^{*}(p)$ (see [4]) is defined by

$$\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)=z^{p}+\sum_{k=1}^{\infty}\frac{(c)_k(p+1-\mu)_k(p+1-\lambda+\nu)_k(\alpha+p)_k}{(a)_k(p+1)_k(p+1-\mu+\nu)_k k!}a_{p+k}(\zeta)z^{p+k}.\qquad(1.2)$$

It is easily verified from (1.2) that

$$z\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z}'=(\alpha+p)\,\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z,\zeta)-\alpha\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta).\qquad(1.3)$$

Differentiating (1.3) $j$-times with respect to $z$, we get

$$z\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^{j+1}}^{(j+1)} = (\alpha+p)\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z,\zeta)\right)_{z^j}^{(j)} - (\alpha+j)\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}. \quad (1.4)$$

Note that the linear operator $\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)$ unifies many other operators considered earlier. In particular:

(1) $\mathcal{L}_{0,\nu}^{0,p,\alpha}(a,c) \equiv \mathcal{J}_p^{\alpha}(a,c)$ (see Cho et al. [1]).
(2) $\mathcal{L}_{0,\nu}^{0,p,\alpha}(a,a) \equiv D^{\alpha+p-1}$ (see Goel and Sohi [2]).
(3) $\mathcal{L}_{0,\nu}^{0,p,1}(p+1-\lambda,1) \equiv \Omega_z^{(\lambda,p)}$ (see Srivastava and Aouf [10]).
(4) $\mathcal{L}_{0,\nu}^{0,1,\alpha-1}(a,c) \equiv \mathcal{J}_c^{a,\alpha}$ (see Hohlov [3]).
(5) $\mathcal{L}_{0,\nu}^{0,1-\alpha,\alpha}(a,c) \equiv \mathcal{L}_p(a,c)$ (see Saitoh [9]).
(6) $\mathcal{L}_{0,\nu}^{0,p,1}(p+\alpha,1) \equiv \mathcal{J}_{\alpha,p}$, $\alpha \in Z$, $\alpha > -p$ (see Liu and Noor [5]).

The purpose of this paper is to apply a method based on the strong differential subordination in order to investigate various useful and interesting properties for higher-order derivatives of multivalent analytic functions involving linear operator.

We need the following lemmas to study the strong differential subordinations:

**Lemma 1.3.** [7] *Let $h(z,\zeta)$ be convex function with $h(0,\zeta)=a$, for every $\zeta \in \overline{U}$ and let $\gamma \in C^* = C\backslash\{0\}$ with $Re(\gamma) \geq 0$. If $p \in \mathcal{H}^*[a,n,\zeta]$ and*

$$p(z,\zeta) + \frac{1}{\gamma} z p_z'(z,\zeta) \prec\prec h(z,\zeta), \ \left(z \in U, \zeta \in \overline{U}\right), \quad (1.5)$$

*then*

$$p(z,\zeta) \prec\prec q(z,\zeta) \prec\prec h(z,\zeta), \ \left(z \in U, \zeta \in \overline{U}\right),$$

*where $q(z,\zeta) = \frac{\gamma}{n z^{\frac{\gamma}{n}}} \int_0^z t^{\frac{\gamma}{n}-1} h(t,\zeta)\,dt$ is convex and it is the best dominant of (1.5).*

**Lemma 1.4.** [6] *Let $q(z,\zeta)$ be convex function in $U \times \overline{U}$ for all $\zeta \in \overline{U}$ and let $h(z,\zeta) = q(z,\zeta) + n\delta z q_z'(z,\zeta)$, $z \in U, \zeta \in \overline{U}$, where $\delta > 0$ and $n$ is a positive integer. If*

$$p(z,\zeta) = q(0,\zeta) + p_n(\zeta) z^n + p_{n+1}(\zeta) z^{n+1} + \cdots,$$

*is analytic in $U \times \overline{U}$ and*

$$p(z,\zeta) + \delta z p_z'(z,\zeta) \prec\prec h(z,\zeta), \ \left(z \in U, \zeta \in \overline{U}\right),$$

*then*

$$p(z,\zeta) \prec\prec q(z,\zeta), \ \left(z \in U, \zeta \in \overline{U}\right),$$

*and this result is sharp.*

## 2. Main results

**Definition 2.1.** Let $\eta \in [0,1)$, $a \in R$, $c \in R \setminus Z_0^-$, $0 \leq \lambda < 1$, $p \in N$, $\alpha > -p$, $\mu, \nu \in R$, $\mu - \nu - p < 1$, $j \in N_0$ and $p > j$. A function $f(z,\zeta) \in \mathcal{A}_\zeta^*(p)$ is said to be in the class $\mathcal{N}(a,c,\lambda,p,\alpha,\mu,\nu,j,\eta,\zeta)$ if it satisfies the inequality

$$Re\left\{\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^{j+1}}^{(j+1)}}{z^{p-j-1}}\right\} > \eta, \quad z \in U, \zeta \in \overline{U}.$$

In the first theorem, we demonstrate that the class $\mathcal{N}(a,c,\lambda,p,\alpha,\mu,\nu,j,\eta,\zeta)$ is convex set.

**Theorem 2.2.** *The set $\mathcal{N}(a,c,\lambda,p,\alpha,\mu,\nu,j,\eta,\zeta)$ is convex.*

*Proof.* Let the functions

$$f_i(z,\zeta) = z^p + \sum_{k=1}^{\infty} a_{p+k,i}(\zeta) z^{p+k}, \ i = 1,2$$

be in the class $\mathcal{N}(a,c,\lambda,p,\alpha,\mu,\nu,j,\eta,\zeta)$. It is sufficient to show that the function $T(z,\zeta) = t_1 f_1(z,\zeta) + t_2 f_2(z,\zeta)$ is in the class $\mathcal{N}(a,c,\lambda,p,\alpha,\mu,\nu,j,\eta,\zeta)$, where $t_1$ and $t_2$ non-negative and $t_1 + t_2 = 1$. Since

$$T(z,\zeta) = z^p + \sum_{k=1}^{\infty} \left(t_1 a_{p+k,1}(\zeta) + t_2 a_{p+k,2}(\zeta)\right) z^{p+k},$$

then

$$\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)T(z,\zeta) =$$
$$z^p + \sum_{k=1}^{\infty} \frac{(c)_k(p+1-\mu)_k(p+1-\lambda+\nu)_k(\alpha+p)_k}{(a)_k(p+1)_k(p+1-\mu+\nu)_k k!} \left(t_1 a_{p+k,1}(\zeta) + t_2 a_{p+k,2}(\zeta)\right) z^{p+k}. \tag{2.1}$$

Differentiating both sides of (2.1) $(j+1)$-times with respect to $z$, we obtain

$$\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)T(z,\zeta)\right)_{z^{j+1}}^{(j+1)} = \frac{p!}{(p-j-1)!} z^{p-j-1} + \sum_{k=1}^{\infty} \frac{(p+k)!}{(p+k-j-1)!} \times$$
$$\times \frac{(c)_k(p+1-\mu)_k(p+1-\lambda+\nu)_k(\alpha+p)_k}{(a)_k(p+1)_k(p+1-\mu+\nu)_k k!} \left(t_1 a_{p+k,1}(\zeta) + t_2 a_{p+k,2}(\zeta)\right) z^{p+k-j-1}.$$

Thus,

$$\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)T(z,\zeta)\right)_{z^{j+1}}^{(j+1)}}{z^{p-j-1}} = \frac{p!}{(p-j-1)!} + \sum_{k=1}^{\infty} \frac{(p+k)!}{(p+k-j-1)!} \times$$
$$\times \frac{(c)_k(p+1-\mu)_k(p+1-\lambda+\nu)_k(\alpha+p)_k}{(a)_k(p+1)_k(p+1-\mu+\nu)_k k!} \left(t_1 a_{p+k,1}(\zeta) + t_2 a_{p+k,2}(\zeta)\right) z^k.$$

Now,

$$Re\left\{\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)T(z,\zeta)\right)_{z^{j+1}}^{(j+1)}}{z^{p-j-1}}\right\}=\frac{p!}{(p-j-1)!}$$
$$+t_1 Re\left\{\sum_{k=1}^{\infty}\frac{(p+k)!}{(p+k-j-1)!}\frac{(c)_k(p+1-\mu)_k(p+1-\lambda+\nu)_k(\alpha+p)_k}{(a)_k(p+1)_k(p+1-\mu+\nu)_k k!}a_{p+k,1}(\zeta)z^k\right\}$$
$$+t_2 Re\left\{\sum_{k=1}^{\infty}\frac{(p+k)!}{(p+k-j-1)!}\frac{(c)_k(p+1-\mu)_k(p+1-\lambda+\nu)_k(\alpha+p)_k}{(a)_k(p+1)_k(p+1-\mu+\nu)_k k!}a_{p+k,2}(\zeta)z^k\right\}. \tag{2.2}$$

Since $f_1, f_2 \in \mathcal{N}(a,c,\lambda,p,\alpha,\mu,\nu,j,\eta,\zeta)$, then we get

$$Re\left\{\sum_{k=1}^{\infty}\frac{(p+k)!}{(p+k-j-1)!}\frac{(c)_k(p+1-\mu)_k(p+1-\lambda+\nu)_k(\alpha+p)_k}{(a)_k(p+1)_k(p+1-\mu+\nu)_k k!}a_{p+k,i}(\zeta)z^k\right\}$$
$$>\eta-\frac{p!}{(p-j-1)!},\quad i=1,2. \tag{2.3}$$

From (2.2) and (2.3), we have

$$Re\left\{\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)T(z,\zeta)\right)_{z^{j+1}}^{(j+1)}}{z^{p-j-1}}\right\}>\frac{p!}{(p-j-1)!}+t_1\left(\eta-\frac{p!}{(p-j-1)!}\right)$$
$$+t_2\left(\eta-\frac{p!}{(p-j-1)!}\right)=\eta.$$

Therefore, $T(z,\zeta) \in \mathcal{N}(a,c,\lambda,p,\alpha,\mu,\nu,j,\eta,\zeta)$ and we obtain the desired result. □

Next, we establish the following inclusion relationship for the class $\mathcal{N}(a,c,\lambda,p,\alpha,\mu,\nu,j,\eta,\zeta)$.

**Theorem 2.3.** *Let* $h(z,\zeta)=\frac{\zeta+(2\eta-\zeta)z}{1+z}$, $z\in U, \zeta\in\overline{U}$, $\eta\in[0,1)$. *Then*

$$\mathcal{N}(a,c,\lambda,p,\alpha+1,\mu,\nu,j,\eta,\zeta)\subset\mathcal{N}(a,c,\lambda,p,\alpha,\mu,\nu,j,\eta^*,\zeta),$$

*where* $\eta^*=2\eta-\zeta+2(\zeta-\eta)(\alpha+p)B(\alpha+p-1)$ *with* $B(x)=\int_0^1\frac{t^x}{1+t}dt$.

*Proof.* Suppose that $f(z,\zeta)\in\mathcal{N}(a,c,\lambda,p,\alpha+1,\mu,\nu,j,\eta,\zeta)$. Using (1.4), we have

$$(\alpha+p)\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}=z\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^{j+1}}^{(j+1)}$$
$$+(\alpha+j)\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}. \tag{2.4}$$

Differentiating both sides of (2.4) with respect to $z$, we get

$$(\alpha+p)\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z,\zeta)\right)_{z^{j+1}}^{(j+1)}=z\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^{j+2}}^{(j+2)}$$
$$+(\alpha+j+1)\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^{j+1}}^{(j+1)}. \tag{2.5}$$

Let the function $F$ be defined by

$$F(z,\zeta)=\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^{j+1}}^{(j+1)}}{z^{p-j-1}},\quad z\in U,\zeta\in\overline{U}.$$

After some computations, we have

$$(p-j-1)\,F(z,\zeta)+zF_z'(z,\zeta)=\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^{j+2}}^{(j+2)}}{z^{p-j-2}}.\tag{2.6}$$

It follows from (2.5) and (2.6) that

$$\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z,\zeta)\right)_{z^{j+1}}^{(j+1)}}{z^{p-j-1}}=F(z,\zeta)+\frac{1}{\alpha+p}zF_z'(z,\zeta).\tag{2.7}$$

Since $f(z,\zeta)\in\mathcal{N}(a,c,\lambda,p,\alpha+1,\mu,\nu,j,\eta,\zeta)$, then from Definition 2.1 and the equation (2.7), we obtain

$$Re\left\{F(z,\zeta)+\frac{1}{\alpha+p}zF_z'(z,\zeta)\right\}>\eta,\quad z\in U,\zeta\in\overline{U}.$$

We note that

$$h_z'(z,\zeta)=\frac{2(\eta-\zeta)}{(1+z)^2}\qquad\text{and}\qquad h_{z^2}''(z,\zeta)=\frac{-4(\eta-\zeta)}{(1+z)^3}.$$

Hence

$$Re\left\{\frac{zh_{z^2}''(z,\zeta)}{h_z'(z,\zeta)}+1\right\}=Re\left\{\frac{1-z}{1+z}\right\}=Re\left\{\frac{1-r(\cos\theta+i\sin\theta)}{1+r(\cos\theta+i\sin\theta)}\right\}$$
$$=\frac{1-r^2}{1+2r\cos\theta+r^2}>0.$$

Therefore $h(z,\zeta)$ is a convex function and

$$F(z,\zeta)+\frac{1}{\alpha+p}zF_z'(z,\zeta)\prec\prec\frac{\zeta+(2\eta-\zeta)z}{1+z}.$$

Since the image of the unit disk $U$ through function $h$ is the semi plane

$$\{w\in C:|w|>\eta\}.$$

By using Lemma 1.3, we deduce that $F(z,\zeta)\prec\prec q(z,\zeta)\prec\prec h(z,\zeta)$, where

$$q(z,\zeta)=\frac{\alpha+p}{z^{\alpha+p}}\int_0^z\frac{\zeta+(2\eta-\zeta)t}{1+t}t^{\alpha+p-1}dt=2\eta-\zeta+2(\zeta-\eta)\frac{\alpha+p}{z^{\alpha+p}}\int_0^z\frac{t^{\alpha+p-1}}{1+t}dt,$$

and $q(z,\zeta)$ is convex and which is the best dominant.

Since $q(z,\zeta)$ is convex and $q\left(U\times\overline{U}\right)$ is symmetric with respect to the real axis, we obtain

$$Re\left\{\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^{j+1}}^{(j+1)}}{z^{p-j-1}}\right\}\geq\min_{|z|=1}Re\{q(z,\zeta)\}$$
$$=Re\{q(1,\zeta)\}$$
$$=2\eta-\zeta+2(\zeta-\eta)(\alpha+p)\int_0^1\frac{t^{\alpha+p-1}}{1+t}dt=\eta^*.$$

Thus, $f(z,\zeta) \in \mathcal{N}(a,c,\lambda,p,\alpha,\mu,\nu,j,\eta^*,\zeta)$ and we obtain the desired result. □

In the next result, we discuss strong subordination property of the integral operator.

**Theorem 2.4.** *Let $q(z,\zeta)$ be a convex function such that $q(0,\zeta)=1$ and let $h$ be the function $h(z,\zeta)=q(z,\zeta)+\frac{1}{\alpha+2p}zq'_z(z,\zeta)$, where $\alpha>-p$. Suppose that*

$$G(z,\zeta)=\frac{\alpha+2p}{z^{\alpha+p}}\int_0^z t^{\alpha+p-1}f(t,\zeta)\,dt, \quad z\in U, \zeta\in\overline{U}. \tag{2.8}$$

*If $f\in\mathcal{A}^*_\zeta(p)$ satisfies the strong differential subordination*

$$\frac{(p-j-1)!}{p!}\frac{\left(\mathcal{L}^{\lambda,p,\alpha}_{\mu,\nu}(a,c)f(z,\zeta)\right)^{(j+1)}_{z^{j+1}}}{z^{p-j-1}}\prec\prec h(z,\zeta), \tag{2.9}$$

*then*

$$\frac{(p-j-1)!}{p!}\frac{\left(\mathcal{L}^{\lambda,p,\alpha}_{\mu,\nu}(a,c)G(z,\zeta)\right)^{(j+1)}_{z^{j+1}}}{z^{p-j-1}}\prec\prec q(z,\zeta)$$

*and this result is sharp.*

*Proof.* Suppose that

$$F(z,\zeta)=\frac{(p-j-1)!}{p!}\frac{\left(\mathcal{L}^{\lambda,p,\alpha}_{\mu,\nu}(a,c)G(z,\zeta)\right)^{(j+1)}_{z^{j+1}}}{z^{p-j-1}}, \quad z\in U, \zeta\in\overline{U}. \tag{2.10}$$

Then the function $F(z,\zeta)$ is analytic in $U\times\overline{U}$ and $F(0,\zeta)=1$.
From (2.8), we have

$$z^{\alpha+p}G(z,\zeta)=(\alpha+2p)\int_0^z t^{\alpha+p-1}f(t,\zeta)\,dt. \tag{2.11}$$

Differentiating both sides of (2.11) with respect to $z$, we get

$$(\alpha+2p)f(z,\zeta)=(\alpha+p)G(z,\zeta)+zG'_z(z,\zeta)$$

and

$$(\alpha+2p)\mathcal{L}^{\lambda,p,\alpha}_{\mu,\nu}(a,c)f(z,\zeta)=(\alpha+p)\mathcal{L}^{\lambda,p,\alpha}_{\mu,\nu}(a,c)G(z,\zeta)+z\left(\mathcal{L}^{\lambda,p,\alpha}_{\mu,\nu}(a,c)G(z,\zeta)\right)'_z.$$

Differentiating the last relation $(j+1)$-times with respect to $z$, we have

$$\left(\mathcal{L}^{\lambda,p,\alpha}_{\mu,\nu}(a,c)f(z,\zeta)\right)^{(j+1)}_{z^{j+1}}=\frac{\alpha+p+j+1}{\alpha+2p}\left(\mathcal{L}^{\lambda,p,\alpha}_{\mu,\nu}(a,c)G(z,\zeta)\right)^{(j+1)}_{z^{j+1}}$$
$$+\frac{z}{\alpha+2p}\left(\mathcal{L}^{\lambda,p,\alpha}_{\mu,\nu}(a,c)G(z,\zeta)\right)^{(j+2)}_{z^{j+2}}.$$

So

$$\frac{(p-j-1)!}{p!}\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f\left(z,\zeta\right)\right)_{z^{j+1}}^{(j+1)}}{z^{p-j-1}}$$
$$=\frac{(\alpha+p+j+1)(p-j-1)!}{(\alpha+2p)p!}\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)G\left(z,\zeta\right)\right)_{z^{j+1}}^{(j+1)}}{z^{p-j-1}}$$
$$+\frac{(p-j-1)!}{(\alpha+2p)p!}\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)G\left(z,\zeta\right)\right)_{z^{j+2}}^{(j+2)}}{z^{p-j-2}}. \tag{2.12}$$

From (2.10) and (2.12), we obtain

$$F\left(z,\zeta\right)+\frac{1}{\alpha+2p}zF_z'\left(z,\zeta\right)=\frac{(p-j-1)!}{p!}\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f\left(z,\zeta\right)\right)_{z^{j+1}}^{(j+1)}}{z^{p-j-1}}. \tag{2.13}$$

Using (2.13), (2.9) becomes

$$F\left(z,\zeta\right)+\frac{1}{\alpha+2p}zF_z'\left(z,\zeta\right)\prec\prec q\left(z,\zeta\right)+\frac{1}{\alpha+2p}zq_z'\left(z,\zeta\right).$$

An application of Lemma 1.4 yields $F\left(z,\zeta\right)\prec\prec q\left(z,\zeta\right)$. By using (2.9), we obtain

$$\frac{(p-j-1)!}{p!}\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)G\left(z,\zeta\right)\right)_{z^{j+1}}^{(j+1)}}{z^{p-j-1}}\prec\prec q\left(z,\zeta\right).$$

□

Next we will give particular case of Theorem 2.4 obtained for appropriate choices of the functions $f(z,\zeta)=\zeta ze^z$ and $q(z,\zeta)=1+\frac{\zeta}{2}z$ with $j=\lambda=\mu=1$, $p=\alpha=c=1$ and $a=2$.

**Example 2.5.** Let $z\in U$, $\zeta\in\overline{U}$. If

$$\zeta(z+1)e^z\prec\prec 1+\frac{2}{3}\zeta z,$$

then

$$3\zeta\left[e^z-\frac{2(z^2-2z+2)e^z+4}{z^3}\right]\prec\prec 1+\frac{\zeta}{2}z.$$

**Theorem 2.6.** *Let $q\left(z,\zeta\right)$ be a convex function such that $q\left(0,\zeta\right)=1$ and let $h$ be the function $h\left(z,\zeta\right)=q\left(z,\zeta\right)+zq_z'\left(z,\zeta\right)$. If $f\in\mathcal{A}_\zeta^*(p)$ satisfies the strong differential subordination*

$$\left(\frac{z\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}}\right)_z'\prec\prec h\left(z,\zeta\right), \tag{2.14}$$

*then*

$$\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}}\prec\prec q\left(z,\zeta\right).$$

*Proof.* Suppose that

$$F(z,\zeta)=\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}},\quad z\in U,\zeta\in\overline{U}. \tag{2.15}$$

Then the function $F(z,\zeta)$ is analytic in $U\times\overline{U}$ and $F(0,\zeta)=1$.
Differentiating both sides of (2.15) with respect to $z$ and using (2.14), we have

$$\begin{aligned}F(z,\zeta)+zF'_z(z,\zeta)&=\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}}\\
&+\frac{z\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z,\zeta)\right)_{z^{j+1}}^{(j+1)}}{\left[\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}\right]^2}\\
&-\frac{z\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^{j+1}}^{(j+1)}}{\left[\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}\right]^2}\\
&=\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}\left(z\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}\right)'_z}{\left[\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}\right]^2}\\
&-\frac{z\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^{j+1}}^{(j+1)}}{\left[\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}\right]^2}\\
&=\left(\frac{z\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}}\right)'_z\prec\prec h(z,\zeta).\end{aligned}$$

An application of Lemma 1.4, we obtain

$$\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha+1}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}}{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}}\prec\prec q(z,\zeta).$$

□

Next we will give particular case of Theorem 2.6 obtained for appropriate choices of the functions $f(z,\zeta)=-\frac{\zeta}{z-2}$ and $q(z,\zeta)=\frac{(2\eta-1)\zeta z+\zeta}{1+z}$, $0\le\eta<1$ with $j=\lambda=\mu=1$, $p=\alpha=c=1$ and $a=2$.

**Example 2.7.** Let $z\in U$, $\zeta\in\overline{U}$. If

$$-\frac{z(z-4)}{2(z-2)^2}\prec\prec\frac{(2\eta-1)\zeta z^2+2\eta\zeta z+\zeta}{(1+z)^2},$$

then

$$-\frac{z}{2\,(z-2)} \prec\prec \frac{(2\eta-1)\zeta z+\zeta}{1+z}.$$

**Theorem 2.8.** *Let $h\,(z,\zeta)$ be a convex function such that $h\,(0,\zeta)=1$. If $f\in\mathcal{A}^*_\zeta(p)$ satisfies the strong differential subordination*

$$\frac{(p-j-1)!}{p!}\frac{\left(\mathcal{L}^{\lambda,p,\alpha}_{\mu,\nu}(a,c)f\,(z,\zeta)\right)^{(j+1)}_{z^{j+1}}}{z^{p-j-1}} \prec\prec h\,(z,\zeta)\,, \tag{2.16}$$

*then*

$$\frac{(p-j)!}{p!}\frac{\left(\mathcal{L}^{\lambda,p,\alpha}_{\mu,\nu}(a,c)f\,(z,\zeta)\right)^{(j)}_{z^{j}}}{z^{p-j}} \prec\prec q\,(z,\zeta) \prec\prec h\,(z,\zeta)\,,$$

*where $q\,(z,\zeta)=\frac{p-j}{z^{p-j}}\int_0^z t^{p-j-1}h\,(t,\zeta)\,dt$ is convex and it is the best dominant.*

*Proof.* Suppose that

$$F\,(z,\zeta)=\frac{(p-j)!}{p!}\frac{\left(\mathcal{L}^{\lambda,p,\alpha}_{\mu,\nu}(a,c)f\,(z,\zeta)\right)^{(j)}_{z^{j}}}{z^{p-j}},\quad z\in U,\zeta\in\overline{U}. \tag{2.17}$$

Then the function $F\,(z,\zeta)$ is analytic in $U\times\overline{U}$ and $F\,(0,\zeta)=1$.
Simple computations from (2.17), we get

$$F\,(z,\zeta)+\frac{1}{p-j}zF'_z\,(z,\zeta)=\frac{(p-j-1)!}{p!}\frac{\left(\mathcal{L}^{\lambda,p,\alpha}_{\mu,\nu}(a,c)f\,(z,\zeta)\right)^{(j+1)}_{z^{j+1}}}{z^{p-j-1}}. \tag{2.18}$$

Using (2.18), (2.16) becomes

$$F\,(z,\zeta)+\frac{1}{p-j}zF'_z\,(z,\zeta)\prec\prec h\,(z,\zeta)\,.$$

An application of Lemma 1.3 with $n=1$, $\gamma=p-j$ yields

$$\frac{(p-j)!}{p!}\frac{\left(\mathcal{L}^{\lambda,p,\alpha}_{\mu,\nu}(a,c)f\,(z,\zeta)\right)^{(j)}_{z^{j}}}{z^{p-j}} \prec\prec q\,(z,\zeta)=\frac{p-j}{z^{p-j}}\int_0^z t^{p-j-1}h\,(t,\zeta)\,dt \prec\prec h\,(z,\zeta)\,.$$

□

By taking $p=1$, $j=0$ and $h\,(z,\zeta)=\frac{\zeta+(2\eta-\zeta)z}{1+z}$, $0\le\eta<1$ in Theorem 2.8, we obtain the following corollary:

**Corollary 2.9.** *If $f\in\mathcal{A}^*_\zeta(p)$ satisfies the strong differential subordination*

$$\left(\mathcal{L}^{\lambda,1,\alpha}_{\mu,\nu}(a,c)f\,(z,\zeta)\right)'_z \prec\prec \frac{\zeta+(2\eta-\zeta)z}{1+z},$$

*then*

$$\frac{\mathcal{L}^{\lambda,1,\alpha}_{\mu,\nu}(a,c)f\,(z,\zeta)}{z} \prec\prec \frac{1}{z}\int_0^z \frac{\zeta+(2\eta-\zeta)t}{1+t}dt=2\eta-\zeta+\frac{2(\zeta-\eta)}{z}\ln(1+z).$$

**Theorem 2.10.** *Let $h(z,\zeta)$ be a convex function such that $h(0,\zeta)=1$. If $f\in\mathcal{A}_{\zeta}^{*}(p)$ satisfies the strong differential subordination*

$$\frac{(1-\beta)(p-j)!}{p!-\sigma(p-j)!}\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}}{z^{p-j}}-\sigma\right)+\frac{\beta(p-j)!}{p!-\sigma(p-j)!}\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^{j+1}}^{(j+1)}}{(p-j)z^{p-j-1}}-\sigma\right)\prec\prec h(z,\zeta), \tag{2.19}$$

*then*

$$\frac{(p-j)!}{p!-\sigma(p-j)!}\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}}{z^{p-j}}-\sigma\right)\prec\prec q(z,\zeta)\prec\prec h(z,\zeta),$$

*where $q(z,\zeta)=\frac{p-j}{\beta}z^{-\frac{p-j}{\beta}}\int_0^z t^{\frac{p-j}{\beta}-1}h(t,\zeta)\,dt$ is convex and it is the best dominant.*

*Proof.* Suppose that

$$F(z,\zeta)=\frac{(p-j)!}{p!-\sigma(p-j)!}\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}}{z^{p-j}}-\sigma\right),\quad z\in U,\zeta\in\overline{U}. \tag{2.20}$$

Then the function $F(z,\zeta)$ is analytic in $U\times\overline{U}$ and $F(0,\zeta)=1$.
Differentiating both sides of (2.20) with respect to $z$, we have

$$F(z,\zeta)+\frac{\beta}{p-j}zF'_z(z,\zeta)=\frac{(1-\beta)(p-j)!}{p!-\sigma(p-j)!}\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}}{z^{p-j}}-\sigma\right)+\frac{\beta(p-j)!}{p!-\sigma(p-j)!}\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^{j+1}}^{(j+1)}}{(p-j)z^{p-j-1}}-\sigma\right). \tag{2.21}$$

From (2.19) and (2.21), we get

$$F(z,\zeta)+\frac{\beta}{p-j}zF'_z(z,\zeta)\prec\prec h(z,\zeta).$$

An application of Lemma 1.3 with $n=1$, $\gamma=\frac{p-j}{\beta}$ yields

$$\frac{(p-j)!}{p!-\sigma(p-j)!}\left(\frac{\left(\mathcal{L}_{\mu,\nu}^{\lambda,p,\alpha}(a,c)f(z,\zeta)\right)_{z^j}^{(j)}}{z^{p-j}}-\sigma\right)\prec\prec q(z,\zeta)=\frac{p-j}{\beta}z^{-\frac{p-j}{\beta}}\int_0^z t^{\frac{p-j}{\beta}-1}h(t,\zeta)\,dt\prec\prec h(z,\zeta).$$

□

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