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STABILITY RESULTS FOR NEUTRAL INTEGRO-DIFFERENTIAL EQUATIONS WITH MULTIPLE FUNCTIONAL DELAYS

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Abstract. Necessary and sufficient conditions for the zero solution of the nonlinear neutral integro-differential equation

$$\frac{d}{dt}\Big(r(t)\Big[x(t)+Q(t,x(t-g_1(t)),...,x(t-g_N(t)))\Big]\Big)$$
$$=-a(t)x(t)+\sum_{i=1}^{N}\int_{t-g_i(t)}^{t}k_i(t,s)f_i(x(s))ds$$

to be asymptotically stable are obtained. In the process we invert the integro-differential equation and obtain an equivalent integral equation. The contraction mapping principle is used as the main mathematical tool for establishing the necessary and sufficient conditions.

1. Introduction and preliminaries

Liapunov's method is normally used to study the stability properties of the zero solution of differential equations. Certain difficulties arise when Liapunov's method is applied to equations with unbounded delay or equations containing unbounded terms [11], [15]. In [6], Burton and Furumochi began a study of stability for ordinary and functional differential equations by means of fixed point theory. They pointed out a number of difficulties encountered in the study of stability by means of Liapunov's direct method. They however noticed that these difficulties frequently vanished when fixed point theory is used instead. In

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the paper they obtained sufficient conditions for the zero solution of the equation

$$x'(t) = -a(t)x(t) + \int_{t-r(t)}^{t} b(t,s)g(x(s))ds$$

to be asymptotically stable.
Raffoul in [15] however obtained sufficient conditions for the asymptotic stability of the zero solution of the neutral delay equation

$$\frac{d}{dt}x(t) = -a(t)x(t) + c(t)\frac{d}{dt}x(t-g(t)) + \int_{t-\tau(t)}^{t} k(t,s)f(x(s))ds$$

using fixed point theory.

In the current paper, we obtain necessary and sufficient conditions for the asymptotic stability of the zero solution of the multiple variable delay neutral integro-differential equation

$$\frac{d}{dt}\Big(r(t)\Big[x(t) + Q(t, x(t-g_1(t)), ..., x(t-g_N(t)))\Big]\Big)$$
$$= -a(t)x(t) + \sum_{i=1}^{N}\int_{t-g_i(t)}^{t} k_i(t,s)f_i(x(s))ds, \tag{1.1}$$

with the initial condition

$$x(t) = \psi(t) \text{ for } t \in [m(t_0), t_0],$$

where $\psi \in C([m(t_0), t_0], \mathbb{R})$ and for each $t_0 \geq 0$,

$$m_j(t_0) = \inf\{t - g_j(t),\ t \geq t_0\},\ m(t_0) = \min\{m_j(t_0),\ 1 \leq j \leq N\}.$$

Neutral differential equations have many applications. For example, they arise in the study of two or more simple oscillatory systems with some interconnections between them [9] and in modeling physical problems such as vibration of masses attached to an elastic bar [17]. Neutral equations also arises in food-limited population models [8] and blood cell models [2]. We refer to [1–10], [12], [13], [15], [18] and [19] for results on stability by fixed piont theory.

The notation $C(S_1, S_2)$ denotes the set of all continuous functions $\varphi : S_1 \to S_2$ with the supremum norm $||.||$. We assume throughout this paper that $a, r \in C(\mathbb{R}^+, \mathbb{R})$, $Q \in C(\mathbb{R}^+ \times \mathbb{R} \times ... \times \mathbb{R}, \mathbb{R})$, $k_j \in C(\mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R})$ $g_j \in C(\mathbb{R}^+, \mathbb{R}^+)$ with $t - g_j(t) \to \infty$ as $t \to \infty$. The function Q is globally Lipschitz continuous in $x_1, x_2, ..., x_N$ and f_j is also globally Lipschitz continuous in x. That is, there are positive constants L_1, $L_2, ..., L_N$ and $\rho_1, \rho_2, ..., \rho_N$ such that

$$|Q(t, x_1, x_2, ..., x_N) - Q(t, y_1, y_2..., y_N)| \leq \sum_{j=1}^{N} L_j||x_j - y_j|| \tag{1.2}$$

and

$$|f_i(x) - f_i(y)| \leq \rho_j||x - y||. \tag{1.3}$$

Moreover, we assume that

$$f_j(0) = 0,\ \text{and}\ Q(t, 0, 0, ..., 0) = 0. \tag{1.4}$$

The rest of the paper is organized as follows. In the next section we state and prove our main result. An example is provided in the last section to illustrate our result.

2. Main results

In this section, we state and prove our main results.

Theorem 2.1. *Suppose (1.2), (1.3) and (1.4) hold. Suppose further that $r(t) \neq 0$ and there exists a constant $\alpha \in (0,1)$ such that for $t \geq 0$*

$$\liminf_{t\to\infty} \int_0^t \frac{r'(s)+a(s)}{r(s)} ds > -\infty, \tag{2.1}$$

and

$$\sum_{j=1}^{N} L_j + \int_{t_0}^{t} \Big(\Big|\frac{a(s)}{r(s)}\Big| (\sum_{j=1}^{N} L_j) + \sum_{i=1}^{N} \rho_i \int_{s-g_i(s)}^{s} \Big|\frac{k_i(s,u)}{r(s)}\Big| du \Big) e^{-\int_s^t \left(\frac{r'(u)+a(u)}{r(u)} du\right)} ds \leq \alpha. \tag{2.2}$$

Then the zero solution of (1.1) is asymptotically stable if and only if

$$\int_0^t \frac{r'(s)+a(s)}{r(s)} ds \to \infty \text{ as } t \to \infty. \tag{2.3}$$

Proof. Suppose first that condition (2.3) hold. Set

$$K = \sup_{t\geq 0} \left\{ e^{-\int_s^t (\frac{r'(u)+a(u)}{r(u)} du)} \right\}, \text{ for } t_0 \geq 0. \tag{2.4}$$

Let the initial function $\psi \in C([m(t_0), t_0], \mathbb{R})$ be fixed and define

$$S = \Big\{ \varphi \in C([m(t_0), \infty), \mathbb{R}) : \varphi(t) \to 0 \text{ as } t \to \infty,\ \varphi(t) = \psi(t) \text{ for } t \in [m(t_0), t_0] \Big\}.$$

We will first show that the set S endowed with the metric $d(x,y) = \sup_{t\geq t_0}\{|x(t)-y(t)|\}$ is a complete metric space. In view of the fact that for every $\varphi \in S$ we have that $\varphi \to 0$ as $t \to \infty$, the functions in S are bounded. Thus, S is a set of bounded functions. We need to show that every Cauchy sequence $\{x_n\}$ of S converges to a bounded function in S.

Given a Cauchy sequence of bounded continuous functions $\{x_n\}$ in S, take some $\tau \in [m(t_0), \infty)$ and consider the sequence of real numbers $\{x_n(\tau)\}$. Given any positive integers m and n, we have

$$|x_m(\tau) - x_n(\tau)| \leq \sup\{|x_m(t) - x_n(t)|; t \in [m(t_0), \infty)\} \equiv d(x_m, x_n). \tag{2.5}$$

Due to the fact that $\{x_n\}$ is a Cauchy sequence, by choosing m and n high enough we can make $|x_m(\tau) - x_n(\tau)|$ arbitrarily small for any τ. Hence, $\{x_n(\tau)\}$ is a Cauchy sequence of real numbers for any τ and because $\mathbb{R}$ is complete with the usual metric, $\{x_n(\tau)\}$ converges to some (finite) real limit, say $x(\tau)$. Next, we establish that the function x is bounded and continuous. To this end we fix an arbitrary point τ in $[m(t_0), \infty)$ and some $\epsilon > 0$. Because $\{x_n\} \to x$, there exists a positive integer N_1 such that $d(x, x_n) < \frac{\epsilon}{3}$ for all $n > N_1$. Hence,

$$|x_n(\tau) - x(\tau)| \leq \sup_t\{|x(t) - x_n(t)|\} \equiv d(x, x_n) < \frac{\epsilon}{3} \tag{2.6}$$

for any t and all $n > N_1$. Moreover, because x_n is continuous, there is some $\delta_1 > 0$ such that for the given τ,

$$|x_n(\tau) - x_n(t)| < \frac{\epsilon}{3} \text{ for all } t \text{ such that } |\tau - t| < \delta_1. \tag{2.7}$$

Thus, choosing $n > N_1$, we have

$$\begin{aligned} |x(\tau) - x(t)| &\leq |x(\tau) - x_n(\tau)| + |x_n(\tau) - x_n(t)| + |x_n(t) - x(t)| \\ &\leq d(x, x_n) + |x_n(\tau) - x_n(t)| + d(x, x_n) < \epsilon. \end{aligned} \tag{2.8}$$

Finally, we show that $d(x, x_n) \to 0$ as $n \to \infty$. Now we fix some $\epsilon > 0$ and observe that because $\{x_n\}$ is Cauchy, there is some N_2 such that

$$d(x_n, x_m) < \frac{\epsilon}{2} \text{ for all } m, n > N_2. \tag{2.9}$$

It follows from (2.9) and the triangle inequality that given any t in $[m(t_0), \infty)$ we have

$$\begin{aligned} |x_n(\tau) - x(\tau)| &\leq |x_n(\tau) - x_m(\tau)| + |x_m(\tau) - x(\tau)| \\ &\leq d(x_n, x_m) + |x_m(\tau) - x(\tau)| \\ &< \frac{\epsilon}{2} + |x_m(\tau) - x(\tau)|, \end{aligned} \tag{2.10}$$

for all $m, n > N_2$. Moreover, because $\{x_m(\tau)\} \to x(\tau)$, we can choose m so that $|x_m(\tau) - x(\tau)| < \frac{\epsilon}{2}$. Hence, N_2 is such that given any $n > N_2$,

$$|x_n(\tau) - x(\tau)| < \epsilon \text{ for all } \tau \text{ in } [m(t_0), \infty).$$

Thus, for n sufficiently high, ϵ is an upper bound for $\{|x_n(\tau) - x(\tau)|;\ \tau \in [m(t_0), \infty)\}$, and because $d(x_n, x)$ is the smallest such upper bound, we conclude that $d(x_n, x) \leq \epsilon$ for all $n > N_2$, that is, $\{x_n\} \to x$. Thus, showing that (S, d) is a complete metric space.

To obtain an appropriate map, we proceed as follows. Rewrite (1.1) as

$$\begin{aligned} &\frac{d}{dt}\Big(x(t) + Q(t, x(t - g_1(t)), ..., x(t - g_N(t)))\Big) \\ &= -\frac{r'(t) + a(t)}{r(t)}(x(t) + Q(t, x(t - g_1(t)), ..., x(t - g_N(t)))) \\ &+ \frac{1}{r(t)}\Big[a(t)Q(t, x(t - g_1(t)), ..., x(t - g_N(t))) \\ &+ \sum_{i=1}^{N} \int_{t-g_i(t)}^{t} k_i(t, s) f_i(x(s)) ds\Big]. \end{aligned} \tag{2.11}$$

Multiplying through (2.11) by $\exp(\int_0^t \left(\frac{r'(s)+a(s)}{r(s)} ds\right)$ gives

$$\begin{aligned}
&\Big[\Big(x(t)+Q(t,x(t-g_1(t)),...,x(t-g_N(t)))\Big)e^{\int_0^t \left(\frac{r'(s)+a(s)}{r(s)} ds\right)}\Big]' \\
&= \frac{1}{r(t)}\Big[a(t)Q(t,x(t-g_1(t)),...,x(t-g_N(t))) \\
&\quad + \sum_{i=1}^N \int_{t-g_i(t)}^t k_i(t,s)f_i(x(s))ds\Big]e^{\int_0^t \left(\frac{r'(s)+a(s)}{r(s)} ds\right)}.
\end{aligned} \tag{2.12}$$

Integrating (2.12) from t_0 to t gives

$$\begin{aligned}
x(t) &= [x(t_0)-Q(t_0,x(t_0-g_1(t_0)),...,x(t_0-g_N(t_0)))]e^{-\int_0^t \left(\frac{r'(s)+a(s)}{r(s)} ds\right)} \\
&\quad - Q(t,x(t-g_1(t)),...,x(t-g_N(t))) \\
&\quad + \int_{t_0}^t \frac{1}{r(s)}\Big[a(s)Q(s,x(s-g_1(s)),...,x(s-g_N(s))) \\
&\quad + \sum_{i=1}^N \int_{s-g_i(s)}^s k_i(s,u)f_i(x(u))du\Big]e^{-\int_s^t \left(\frac{r'(u)+a(u)}{r(u)} ds\right)} ds.
\end{aligned} \tag{2.13}$$

Thus, we define the operator $P : S \to S$ by $(P\varphi)(t) = \psi(t)$ for $t \in [m(t_0), t_0]$ and

$$\begin{aligned}
(P\varphi)(t) &= [\psi(t_0)-Q(t_0,x(t_0-g_1(t_0)),...,x(t_0-g_N(t_0)))]e^{-\int_0^t \left(\frac{r'(s)+a(s)}{r(s)} ds\right)} \\
&\quad - Q(t,x(t-g_1(t)),...,x(t-g_N(t))) \\
&\quad + \int_{t_0}^t \frac{1}{r(s)}\Big[a(s)Q(s,x(s-g_1(s)),...,x(s-g_N(s))) \\
&\quad + \sum_{i=1}^N \int_{s-g_i(s)}^s k_i(s,u)f_i(x(u))du\Big]e^{-\int_s^t \left(\frac{r'(u)+a(u)}{r(u)} ds\right)} ds \text{ for } t \geq t_0.
\end{aligned} \tag{2.14}$$

It is not difficult to see that $(P\varphi)(t)$ is continuous. We next show that $(P\varphi)(t) \to 0$ as $t \to \infty$. The first term on the right hand side of (2.14) goes to zero because of condition (2.1). Since $t - g_j(t) \to \infty$, $j = 1, 2, ..., N$, as $t \to \infty$, and the fact that $\varphi \in S$, $Q(t, \varphi(t - g_1(t)), ..., \varphi(t - g_N(t))) \to Q(t, 0, ..., 0)$ as $t \to \infty$. Thus, showing that the second term on the right hand side of (2.14) goes to zero as $t \to \infty$.

Now we show that the last term on the right hand side of (2.14) goes to zero as $t \to \infty$. Since $\varphi(t) \to 0$ and $t - g_j(t)$, $j = 1, 2, ..., N$, as $t \to \infty$, for each $\epsilon_1 > 0$, there exists a $T_1 > t_0$ such that $s \geq T_1$ implies $|\varphi(s - g_j(t))| < \epsilon_1$ for $j = 1, ..., N$.

Thus for $t \geq T_1$, the last term, I_3 in (2.14) satisfies

$$\begin{aligned}
|I_3| &= \Big| \int_{t_0}^{t} \frac{1}{r(s)} \Big[a(s) Q(s, x(s-g_1(s)), ..., x(s-g_N(s)))) \\
&\quad + \sum_{i=1}^{N} \int_{s-g_i(s)}^{s} k_i(s,u) f_i(x(u)) du \Big] e^{-\int_s^t \left(\frac{r'(u)+a(u)}{r(u)} ds \right)} ds \Big| \\
&\leq \int_{t_0}^{T_1} |\frac{1}{r(s)}| \Big[|a(s)||Q(s, x(s-g_1(s)), ..., x(s-g_N(s))))| \\
&\quad + |\sum_{i=1}^{N} \int_{s-g_i(s)}^{s} k_i(s,u) f_i(x(u)) du| \Big] e^{-\int_s^t \left(\frac{r'(u)+a(u)}{r(u)} ds \right)} ds \\
&\quad + \int_{T_1}^{t} |\frac{1}{r(s)}| \Big[|a(s)||Q(s, x(s-g_1(s)), ..., x(s-g_N(s))))| \\
&\quad + |\sum_{i=1}^{N} \int_{s-g_i(s)}^{s} k_i(s,u) f_i(x(u)) du| \Big] e^{-\int_s^t \left(\frac{r'(u)+a(u)}{r(u)} ds \right)} ds \leq
\end{aligned}$$

$$\sup_{\sigma \geq m(t_0)} |\varphi(\sigma)|$$

$$\times \int_{t_0}^{T_1} \frac{1}{|r(s)|} \Big(|a(s)| (\sum_{j=1}^{N} L_j) + \sum_{i=1}^{N} \rho_i \int_{s-g_i(s)}^{s} |k_i(s,u)| du \Big) e^{-\int_s^t \left(\frac{r'(u)+a(u)}{r(u)} du \right)} ds$$

$$+\epsilon_1 \int_{T_1}^{t} \frac{1}{|r(s)|} \Big(|a(s)| (\sum_{j=1}^{N} L_j) + \sum_{i=1}^{N} \rho_i \int_{s-g_i(s)}^{s} |k_i(s,u)| du \Big) e^{-\int_s^t \left(\frac{r'(u)+a(u)}{r(u)} du \right)} ds$$

$$\leq \epsilon_1 + \alpha \epsilon_1 < 2\epsilon_1.$$

Thus, showing that $I_3 \to 0$ as $t \to \infty$. This yields $(P\varphi)(t) \to 0$ as $t \to \infty$, and hence $P\varphi \in S$. We next show that P is a contraction. To this end, let $\varphi, \eta \in S$. Then

$$\begin{aligned}
|(P\varphi)(t) - (P\eta)(t)| &\leq \Big(\sum_{j=1}^{N} L_j + \int_{t_0}^{t} \frac{1}{|r(s)|} \Big(|a(s)| (\sum_{j=1}^{N} L_j) \\
&\quad + \sum_{i=1}^{N} \rho_i \int_{s-g_i(s)}^{s} |k_i(s,u)| du \Big) e^{-\int_s^t \left(\frac{r'(u)+a(u)}{r(u)} du \right)} ds \Big) \\
&\quad \times \|\varphi - \eta\| \\
&\leq \alpha \|\varphi - \eta\|.
\end{aligned}$$

Thus, by the contraction mapping principle, P has a unique fixed point x in S which is a solution of (1.1) with $x(t) = \psi(t)$ on $[m(t_0), t_0]$ and $x(t) = x(t, t_0, \psi) \to 0$ as $t \to \infty$.

For the zero solution of (1.1) to be asymptotically stable, we need to show that the zero solution of (1.1) is stable. To accomplish this, let $\epsilon > 0$ be given and choose $\delta > 0$ $(\delta < \epsilon)$ satisfying $2\delta K e^{\int_0^{t_0}\left(\frac{r'(u)+a(u)}{r(u)}ds\right)} + \alpha\epsilon < \epsilon$. If $x(t) = x(t, t_0, \psi)$ is a solution of (1.1) with $||\psi|| < \delta$, then $x(t) = (Px)(t)$ defined by (2.14). We claim that $|x(t)| < \epsilon$ for all $t > t_0$. Observe that $|x(s)| < \epsilon$ on $[m(t_0), t_0]$. If there exists $t^* > t_0$ such that $|x(t^*)| = \epsilon$ and $|x(s)| < \epsilon$ for $m(t_0) \le s < t^*$, then it follows from (2.14) that

$$\begin{aligned} |x(t^*)| &\le ||\psi||[1 + \sum_{j=1}^{N} L_j] e^{-\int_0^{t^*}\left(\frac{r'(s)+a(s)}{r(s)}ds\right)} + \epsilon \sum_{j=1}^{N} L_j \\ &\quad + \epsilon \int_{t_0}^{t^*} |\frac{1}{|r(s)|}\Big(|a(s)|(\sum_{j=1}^{N} L_j) + \sum_{i=1}^{N} \rho_i \int_{s-g_i(s)}^{s} |k_i(s,u)|du\Big) \\ &\qquad \times e^{-\int_s^t\left(\frac{r'(u)+a(u)}{r(u)}ds\right)} ds \\ &\le 2\delta K e^{\int_0^{t_0}\left(\frac{r'(u)+a(u)}{r(u)}ds\right)} + \alpha\epsilon < \epsilon, \end{aligned} \tag{2.15}$$

which contradicts the definition of t^*. Thus, $|x(t)| < \epsilon$ for all $t \ge t_0$, and the zero solution of (1.1) is stable. This shows that the zero solution of (1.1) is asymptotically stable if (2.3) holds.

Conversely, suppose (2.3) fails. Then by (2.1) there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_0^{t_n} \frac{r'(s)+a(s)}{r(s)} ds = l$ for some $l \in \mathbb{R}$. We may also choose a positive constant J satisfying

$$-J \le \int_0^{t_n} \frac{r'(s)+a(s)}{r(s)} ds \le J,$$

for all $n \ge 1$. But, in view of condition (2.2) we have

$$\begin{aligned} &\int_0^{t_n} \frac{1}{|r(s)|}\Big(|a(s)|(\sum_{j=1}^{N} L_j) \\ &+ \sum_{i=1}^{N} \rho_i \int_{s-g_i(s)}^{s} |k_i(s,u)|du\Big) e^{-\int_s^t\left(\frac{r'(u)+a(u)}{r(u)}du\right)} ds \le \alpha. \end{aligned}$$

This yields

$$\begin{aligned} &\int_0^{t_n} \frac{1}{|r(s)|}\Big(|a(s)|(\sum_{j=1}^{N} L_j) \\ &+ \sum_{i=1}^{N} \rho_i \int_{s-g_i(s)}^{s} |k_i(s,u)|du\Big) e^{\int_0^s\left(\frac{r'(u)+a(u)}{r(u)}du\right)} ds \le \alpha e^{\int_0^{t_n}\left(\frac{r'(u)+a(u)}{r(u)}du\right)} \le e^J. \end{aligned}$$

Thus, the sequence

$$\Big\{ \int_0^{t_n} \frac{1}{|r(s)|}\Big(|a(s)|(\sum_{j=1}^{N} L_j) + \sum_{i=1}^{N} \rho_i \int_{s-g_i(s)}^{s} |k_i(s,u)|du\Big) e^{\int_0^s\left(\frac{r'(u)+a(u)}{r(u)}du\right)} ds \Big\}$$

is bounded, so there exists a convergent subsequence. Suppose

$$\lim_{n\to\infty}\int_0^{t_n}\frac{1}{|r(s)|}\Big(|a(s)|(\sum_{j=1}^{N}L_j) + \sum_{i=1}^{N}\rho_i\int_{s-g_i(s)}^{s}|k_i(s,u)|du\Big)e^{\int_0^s\left(\frac{r'(u)+a(u)}{r(u)}du\right)}ds = \mu,$$

for some $\mu \in \mathbb{R}^+$ and choose a positive integer i so large that

$$\int_{t_i}^{t_n}\frac{1}{|r(s)|}\Big(|a(s)|(\sum_{j=1}^{N}L_j) + \sum_{i=1}^{N}\rho_i\int_{s-g_i(s)}^{s}|k_i(s,u)|du\Big)e^{\int_0^s\left(\frac{r'(u)+a(u)}{r(u)}du\right)}ds < \delta_0/4K$$

for all $n \geq i$, where $\delta_0 > 0$ satisfies $4\delta_0 K e^J + \alpha < 1$. Now, consider the solution $x(t) = x(t, t_i, \psi)$ of (1.1) with $\psi(t_i) = \delta_0$ and $|\psi(s)| \leq \delta_0$ for $s \leq t_i$. A reasoning similar to that in (2.15) gives $|x(t)| \leq 1$ for $t \geq t_i$. We may choose ψ so that

$$\psi(t_i) - Q(t_i, \psi(t_i - g_1(t_i)), ..., \psi(t_i - g_N(t_i))) \geq \frac{1}{2}\delta_0. \tag{2.16}$$

It follows from (2.14) with $x(t) = (Px)(t)$ that for $n \geq t_i$,

$$\begin{aligned}
&\Big|x(t_n) - Q(t_n, x(t_n - g_1(t_n)), ..., x(t_n - g_N(t_n)))\Big| \\
&\geq \frac{1}{2}\delta_0 e^{-\int_{t_i}^{t_n}\left(\frac{r'(u)+a(u)}{r(u)}du\right)}ds \\
&- \int_{t_i}^{t_n}\frac{1}{|r(s)|}\Big(|a(s)|(\sum_{j=1}^{N}L_j) + \sum_{i=1}^{N}\rho_i\int_{s-g_i(s)}^{s}|k_i(s,u)|du\Big)e^{-\int_s^{t_n}\left(\frac{r'(u)+a(u)}{r(u)}du\right)}ds \\
&= \frac{1}{2}\delta_0 e^{-\int_{t_i}^{t_n}\left(\frac{r'(u)+a(u)}{r(u)}du\right)} \\
&- e^{-\int_0^{t_n}\left(\frac{r'(u)+a(u)}{r(u)}du\right)}\int_{t_i}^{t_n}\frac{1}{|r(s)|}\Big(|a(s)|(\sum_{j=1}^{N}L_j) + \sum_{i=1}^{N}\rho_i\int_{s-g_i(s)}^{s}|k_i(s,u)|du\Big) \\
&\times e^{\int_0^s\left(\frac{r'(u)+a(u)}{r(u)}du\right)}ds \\
&= e^{-\int_{t_i}^{t_n}\left(\frac{r'(u)+a(u)}{r(u)}du\right)}\Big[\frac{1}{2}\delta_0 - e^{-\int_0^{t_i}\left(\frac{r'(u)+a(u)}{r(u)}du\right)} \\
&\times \int_{t_i}^{t_n}\frac{1}{|r(s)|}\Big(|a(s)|(\sum_{j=1}^{N}L_j) + \sum_{i=1}^{N}\rho_i\int_{s-g_i(s)}^{s}|k_i(s,u)|du\Big)e^{\int_0^s\left(\frac{r'(u)+a(u)}{r(u)}du\right)}ds\Big] \\
&\geq e^{-\int_{t_i}^{t_n}\left(\frac{r'(u)+a(u)}{r(u)}du\right)}\Big[\frac{1}{2}\delta_0
\end{aligned}$$

$$-K\int_{t_i}^{t_n}\frac{1}{|r(s)|}\Big(|a(s)|(\sum_{j=1}^{N}L_j)+\sum_{i=1}^{N}\rho_i\int_{s-g_i(s)}^{s}|k_i(s,u)|du\Big)e^{\int_0^s\left(\frac{r'(u)+a(u)}{r(u)}du\right)}ds\Big]$$

$$\geq \frac{1}{4}\delta_0 e^{-\int_{t_i}^{t_n}\left(\frac{r'(u)+a(u)}{r(u)}du\right)}\geq\frac{1}{4}\delta_0 e^{-2J}>0. \qquad (2.17)$$

On the other hand, if the zero solution of (1.1) is asymptotically stable, then $x(t)=x(t,t_i,\psi)\to 0$ as $t\to\infty$. Since $t_n-g(t)\to\infty$ as $n\to\infty$ and (2.2) holds, we have

$$x(t_n)-Q(t_n,x(t_n-g_1(t_n)),...,x(t_n-g_N(t_n)))\to 0 \text{ as } t\to\infty,$$

which contradicts (2.17). Hence, condition (2.3) is necessary for the asymptotic stability of the zero solution of (1.1). ☐

3. Example

In this section we provide an example to illustrate our main result. Consider the equation

$$\frac{d}{dt}\Big((t^2+t+1)\Big[x(t)+\frac{1}{200}x(t-\frac{t}{3})+\frac{1}{400}x(t-\frac{t}{2})\Big]\Big) \qquad (3.1)$$

$$=-(2t+1)x(t)+\int_{t-\frac{t}{2}}^{t}\frac{2}{(t^2+t+1)^2}f_1(x(s))ds$$

$$+\int_{t-\frac{t}{3}}^{t}\frac{1+2s}{(\frac{t}{3}+\frac{5t^2}{9})(t^2+t+1)}f_2(x(s))ds$$

where

$$f_1(x(t))=\frac{1}{50}x(t) \text{ and } f_2(x(t))=\frac{1}{100}x(t).$$

Thus,

$$\begin{aligned}\int_0^t\frac{r'(s)+a(s)}{r(s)}ds &= \int_0^t\frac{2(2s+1)}{s^2+s+1}ds\\ &= 2\ln(t^2+t+1)\to\infty \text{ as } t\to\infty.\end{aligned}$$

Also,

$$\liminf_{t\to\infty}\Big(2\ln(t^2+t+1)\Big)>-\infty,$$

Moreover,

$$e^{-\int_s^t\left(\frac{r'(u)+a(u)}{r(u)}du\right)}=\frac{(s^2+s+1)^2}{(t^2+t+1)^2}$$

Thus,

$$\int_{t_0}^{t}\Big|\frac{a(s)}{r(s)}\Big|(\sum_{j=1}^{N}L_j)e^{-\int_s^t\left(\frac{r'(u)+a(u)}{r(u)}du\right)}ds$$
$$=\frac{3}{400(t^2+t+1)^2}\int_0^t(2s+1)(s^2+s+1)ds\le\frac{3}{400}$$

$$\rho_1\int_{t_0}^{t}\Big(\int_{s-g_1(s)}^{s}\Big|\frac{k_1(s,u)}{r(s)}\Big|du\Big)e^{-\int_s^t\left(\frac{r'(u)+a(u)}{r(u)}du\right)}ds$$
$$\le\frac{2}{50(t^2+t+1)^2}\int_0^t\int_{s-\frac{s}{2}}^{s}duds\le\frac{2}{50}$$

and

$$\rho_2\int_{t_0}^{t}\Big(\int_{s-g_1(s)}^{s}\Big|\frac{k_2(s,u)}{r(s)}\Big|du\Big)e^{-\int_s^t\left(\frac{r'(u)+a(u)}{r(u)}du\right)}ds$$
$$\le\frac{1}{100(t^2+t+1)^2}\int_0^t\int_{s-\frac{s}{3}}^{s}\frac{1+2u}{(\frac{s}{3}+\frac{5s^2}{9})}duds\le\frac{1}{100}.$$

It therefore follows that

$$\sum_{j=1}^{N}L_j+\int_{t_0}^{t}\Big(\Big|\frac{a(s)}{r(s)}\Big|(\sum_{j=1}^{N}L_j)$$
$$+\sum_{i=1}^{N}\rho_i\int_{s-g_i(s)}^{s}\Big|\frac{k_i(s,u)}{r(s)}\Big|du\Big)e^{-\int_s^t\left(\frac{r'(u)+a(u)}{r(u)}du\right)}ds$$
$$\le\ \frac{26}{400}<1.$$

Hence, all the conditions in Theorem 2.1 are satisfied. Therefore, the zero solution of equation (3.1) is asymptotically stable.

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