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ON A CONTROL DYNAMICAL SYSTEM WITH MAXIMAL MONOTONE OPERATORS

NADJIBA ABDI[1] AND SOUMIA SAÏDI[2*]

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Abstract. This paper studies a minimizing problem subject to a control dynamical system involving two differential inclusions driven by maximal monotone operators and an integral perturbation. First, an existence result, for a mixed partially bounded variation continuous differential system, is obtained via a discretization scheme (in the context of Hilbert spaces). The latter permits us to deduce the well-posedness of the control dynamical system under consideration. Finally, under suitable assumptions on the sets of control maps acting in both the state of the operators and the time-variables of the perturbations, the optimality result is proved.

1. Introduction and background material

We will continue, in this paper, the study begun in the recent contribution [28] regarding a class of dynamical systems involving differential inclusions with maximal monotone operators. Our main concern is to deal with the dynamical system proposed in the perspectives of [28]. The first-order mixed partially bounded variation continuous (BVC shortly) differential system associated to ρ to be investigated here, is

$$\begin{cases} -\dfrac{du}{d\rho}(t) \in B_1(t)u(t) + \displaystyle\int_0^t f_1(t,s,x(s),u(s))d\rho(s)\; d\rho-\text{a.e. } t \in I := [0,T], \\ -\ \dot{x}(t) \in B_2(t)x(t) + f_2(t,x(t),u(t)) \quad \text{a.e. } t \in I, \\ \ (u(0),x(0)) = (u_0,x_0) \in \mathrm{D}\,(B_1(0)) \times \mathrm{D}\,(B_2(0)), \end{cases} \tag{1.1}$$

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* Corresponding author.
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where H is a real separable Hilbert space and $\rho : I \to [0, +\infty[$ is a continuous nondecreasing map on I. The operators $B_1(t) : \mathrm{D}(B_1(t)) \subset H \to 2^H$ and $B_2(t) : \mathrm{D}(B_2(t)) \subset H \to 2^H$ are maximal monotone with domains denoted by $\mathrm{D}(B_1(t))$ and $\mathrm{D}(B_2(t))$, respectively, for each $t \in I$. The dependence $t \mapsto B_1(t)$ (resp., $t \mapsto B_2(t)$) is BVC (resp., absolutely continuous) on I with respect to the pseudo-distance. The perturbations $f_1 : I \times I \times H \times H \to H$ and $f_2 : I \times H \times H \to H$ are single-valued maps.

The first novelty of our contribution is that we mix two differential inclusions involving maximal monotones operators such that a density with respect to $d\rho$ ($\rho(\cdot)$ is a bounded continuous map) is taken in the first-one, while the derivative in the second-one is taken with respect to the Lebesgue measure, with the introduction of an integral perturbation in the new system.

Let us stress that the dynamical system (1.1) cannot be reduced to one (only) differential inclusion formulated by

$$\begin{cases} -\dfrac{dV}{d\mu}(t) \in B(t)V(t) + h(t, V(t)) \quad d\mu\text{−a.e.} \qquad t \in I, \\ V(t) \in \mathrm{D}(B(t)), \quad t \in I, \\ V(0) = (u_0, x_0) \in \mathrm{D}(B(0)), \end{cases}$$

by finding a time-dependent maximal monotone operator $B(t)$, $d\mu = dt + d\rho$, and a suitable perturbation $h(\cdot, \cdot)$. In the current situation, it is not possible to apply the result in [3]. So, we proceed via a discretization method, by proving the convergence of the approximate solutions $(u_n, x_n)_n$ to the solution (u, x) of the original system (1.1).

Recent developments on dynamical systems with two first-order differential inclusions governed by maximal monotone operators or sweeping processes or subdifferentials have occurred in numerous papers; see for instance [2, 7, 14, 23, 24, 26, 28, 29], and those on differential inclusions with integral perturbations can be found in, for example, [8, 9, 11, 17, 18, 22].

Our second topic is motivated by the recent study in [28, Theorem 5.2], concerning a minimization problem over the solution set of a dynamical system involving maximal monotone operators, where the control maps act only on the single-valued perturbations. Also, we are inspired by [27, Theorem 3.3], regarding an optimization problem subject to a differential inclusion with maximal monotone operators, where the control maps act only in the state of the operators. We investigate here optimal solutions to the following problem:

$$\min_{(y,z) \in \mathcal{Y} \times \mathcal{Z}} \varphi(u_{y,z}(T), x_{y,z}(T)),$$

where $\varphi : H \times H \to \mathbb{R}$ is lower semi-continuous, $\mathcal{Y}_i$ and $\mathcal{Z}_i$ are suitable sets ($i = 1, 2$), and $(u_{y,z}, x_{y,z})$ is the unique solution associated to the controls y, z of the following dynamical system:

$$\begin{cases} -\dot{u}(t) \in B_1(t, y_1(t))u(t) + \displaystyle\int_0^t f_1(t, s, x(s), u(s), z_1(s))ds \quad \text{a.e. } t \in I, \\ -\dot{x}(t) \in B_2(t, y_2(t))x(t) + f_2(t, x(t), u(t), z_2(t)) \quad \text{a.e. } t \in I, \\ (y_0^1, y_0^2) = (y_1(0), y_2(0)), \\ z = (z_1, z_2) \in \mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2, \\ y = (y_1, y_2) \in \mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2, \\ (u(0), x(0)) = (u_0, x_0) \in \mathrm{D}\,(B_1(0, y_0^1)) \times \mathrm{D}\,(B_2(0, y_0^2)). \end{cases} \tag{1.2}$$

The second novelty in the present work is that, in the minimization problem subject to the dynamical system, it is considered controls acting in both the state of the operators and the perturbations.

Let us point out that optimal control of systems driven by ordinary differential equations with nonlinear differential inclusions have been investigated in the scientific literature; see, for example, [1, 10, 15], among others. Sweeping processes (or differential inclusions governed by maximal monotone operators) involving control actions and optimization have been discussed also; see, for example, [5, 6, 13, 16, 19, 20, 21], and the references therein.

The article consists of three sections. After recalling some basic notations, definitions, and background material needed in our development, our main existence result concerning the dynamical system (1.1) is proved in the next section, using a discretization method. The last section is dedicated to the study of the minimization problem above.

Now, we recall the basic notations needed in what follows.

Let H be a real separable Hilbert space whose inner product is denoted by $\langle \cdot, \cdot \rangle$ and its associated norm $\| \cdot \|$. Denote by $\overline{B}_H$ the closed unit ball of H and by $\overline{B}_H[u_0, L]$ the closed ball of center u_0 and radius L.
Given an interval $I := [0, T]$ $(T > 0)$ of $\mathbb{R}$, let $\mathcal{C}(I, H)$ be the space of continuous maps $u : I \to H$, endowed with the norm of uniform convergence on I: $\|u\|_\infty = \sup_{t \in I} \|u(t)\|$.
Let $L^p(I, H)$ for $p \in [1, +\infty[$ (resp., $p = +\infty$), be the space of measurable maps $u : I \to H$ such that $\int_I \|u(t)\|^p dt < +\infty$ (resp., which are essentially bounded) endowed with the usual norm $\|u\|_{L^p(I,H)} = (\int_I \|u(t)\|^p dt)^{\frac{1}{p}}$, $1 \le p < +\infty$ (resp., endowed with the usual essential supremum norm $\| \cdot \|_{L^\infty(I,H)}$). By $W^{1,2}(I, H)$, we denote the space of absolutely continuous functions from I to H with derivatives in $L^2(I, H)$.

Let us summarize some properties of maximal monotone operators. Let $B : D(B) \subset H \to 2^H$ be a set-valued operator whose domain, range, and graph are defined by

$$\begin{aligned} \mathrm{D}\,(B) &= \{u \in H : \ Bu \neq \emptyset\}, \\ R(B) &= \{v \in H : \ \text{there exists } u \in \mathrm{D}\,(B), \ v \in Bu\} = \cup\{Bu : \ u \in \mathrm{D}\,(B)\}, \\ Gr(B) &= \{(u, v) \in H \times H : \ u \in \mathrm{D}\,(B), \ v \in Bu\}. \end{aligned}$$

Definition 1.1. [12] The operator $B : D(B) \subset H \to 2^H$ is said to be monotone if $\langle v_1 - v_2, u_1 - u_2\rangle \geq 0$ whenever $(u_i, v_i) \in Gr(B)$, $i = 1, 2$. It is maximal monotone if its graph could not be contained strictly in the graph of any other monotone operator. In this case, for all $\mu > 0$, $R(I_H + \mu B) = H$, where I_H denotes the identity map of H.

Proposition 1.2. [4] *If B is a maximal monotone operator, then, for every $u \in \mathrm{D}(B)$, Bu is nonempty, closed, and convex. Moreover, the projection of the origin onto Bu, $B^0(u)$ exists and is unique.*

Definition 1.3. [12] For $\mu > 0$, we define the resolvent and the Yosida approximation of B, respectively, by $J^B_\mu = (I_H + \mu B)^{-1}$ and $B_\mu = \frac{1}{\mu}\left(I_H - J^B_\mu\right)$.

Proposition 1.4. [4] *Both operators J^B_μ and B_μ are single-valued and defined on the whole space H, and one has*

$$J^B_\mu u \in \mathrm{D}(B) \ \text{ and } \ B_\mu(u) \in B(J^B_\mu u) \text{ for every } u \in H, \tag{1.3}$$

$$\|B_\mu(u)\| \leq \|B^0(u)\| \ \text{ for every } u \in \mathrm{D}(B).$$

We recall now the definition of the pseudo-distance between two maximal monotone operators.

Definition 1.5. [30] Let $B_1 : D(B_1) \subset H \to 2^H$ and $B_2 : D(B_2) \subset H \to 2^H$ be two maximal monotone operators. Then we denote by $\mathrm{dis}\,(B_1, B_2)$ the pseudo-distance between B_1 and B_2 defined by

$$\mathrm{dis}\,(B_1, B_2) = \sup\left\{\frac{\langle v_1 - v_2, u_2 - u_1\rangle}{1 + \|v_1\| + \|v_2\|} : \ (u_1, v_1) \in Gr(B_1),\ (u_2, v_2) \in Gr(B_2)\right\}.$$

Remark 1.6. Observe that $\mathrm{dis}\,(B_1, B_2) \in [0, +\infty]$, $\mathrm{dis}\,(B_1, B_2) = \mathrm{dis}\,(B_2, B_1)$ and $\mathrm{dis}\,(B_1, B_2) = 0$ if and only if $B_1 = B_2$.

Lemma 1.7. [3] *For any nonnegative real number μ, one has*

$$\mathrm{dis}\,(\mu B_1, \mu B_2) \leq \max\{1, \mu\}\mathrm{dis}\,(B_1, B_2). \tag{1.4}$$

Let us recall some useful lemmas.

Lemma 1.8. [25] *Let B be a maximal monotone operator of H. If $u \in \overline{\mathrm{D}(B)}$ and $v \in H$ are such that*

$$\langle B^0 w - v, w - u\rangle \geq 0 \quad \text{for all } w \in \mathrm{D}(B),$$

then $u \in \mathrm{D}(B)$ and $v \in Bu$.

Lemma 1.9. [25] *Let B_n $(n \in \mathbb{N})$ and B be maximal monotone operators of H such that $\mathrm{dis}\,(B_n, B) \to 0$. Suppose also that $u_n \in \mathrm{D}(B_n)$ with $u_n \to u$ and $v_n \in B_n u_n$ with $v_n \to v$ weakly for some $u, v \in H$. Then, $u \in \mathrm{D}(B)$ and $v \in Bu$.*

Lemma 1.10. [25] *Let B_1, B_2 be maximal monotone operators of H. Then,*
(1) *for $\mu > 0$ and $u \in \mathrm{D}(B_1)$*

$$\|u - J^{B_2}_\mu(u)\| \leq \mu\|B^0_1(u)\| + \mathrm{dis}\,(B_1, B_2) + \sqrt{\mu\big(1 + \|B^0_1 u\|\big)\mathrm{dis}\,(B_1, B_2)};$$

(2) *for* $\mu > 0$ *and* $u, v \in H$

$$\|J_\mu^{B_1}(u) - J_\mu^{B_1}(v)\| \leq \|u - v\|.$$

Lemma 1.11. [25] *Let* B_n $(n \in \mathbb{N})$ *and* B *be maximal monotone operators of* H *such that* $\operatorname{dis}(B_n, B) \to 0$ *and* $\|B_n^0 u\| \leq c(1 + \|u\|)$ *for some* $c > 0$, *all* $n \in \mathbb{N}$ *and* $u \in \mathrm{D}(B_n)$. *Then, for every* $w \in \mathrm{D}(B)$, *there exists a sequence* (w_n) *such that*

$$w_n \in \mathrm{D}(B_n), \quad w_n \to w \ \text{ and } \ B_n^0 w_n \to B^0 w.$$

The discrete version of Gronwall's lemma is given as follows.

Lemma 1.12. [25] *Let* $\alpha > 0$. *Let* (γ_i) *and* (η_i) *be sequences of nonnegative real numbers such that*

$$\eta_{i+1} \leq \alpha + (\sum_{k=0}^{i} \gamma_k \eta_k) \quad \text{for } i \in \mathbb{N}^*.$$

Then, one has

$$\eta_{i+1} \leq \alpha \exp(\sum_{k=0}^{i} \gamma_k) \quad \text{for } i \in \mathbb{N}^*.$$

To close this section, recall the following Gronwall-like differential inequality.

Lemma 1.13. [9] *Let* $v : [T_0, T] \to \mathbb{R}$ *be a nonnegative absolutely continuous function and let* $h_1, h_2, g : [T_0, T] \to \mathbb{R}_+$ *be nonnegative integrable functions. Suppose for some* $\varepsilon > 0$

$$\dot{v}(t) \leq g(t) + \varepsilon + h_1(t)v(t) + h_2(t)(v(t))^{\frac{1}{2}} \int_{T_0}^{t} (v(s))^{\frac{1}{2}} ds \text{ a.e. } t \in [T_0, T].$$

Then, for all $t \in [T_0, T]$, *one has*

$$\begin{aligned}
(v(t))^{\frac{1}{2}} \leq &(v(T_0) + \varepsilon)^{\frac{1}{2}} \exp\left(\int_{T_0}^{t} (h(s) + 1) ds\right) + \frac{\varepsilon^{\frac{1}{2}}}{2} \int_{T_0}^{t} \exp\left(\int_{s}^{t} (h(r) + 1) d\rho\right) ds \\
&+ 2\left[\left(\int_{T_0}^{t} g(s) ds + \varepsilon\right)^{\frac{1}{2}} - \varepsilon^{\frac{1}{2}} \exp\left(\int_{T_0}^{t} (h(r) + 1) d\rho\right)\right] \\
&+ 2 \int_{T_0}^{t} \left(h(s) + 1\right) \exp\left(\int_{s}^{t} (h(r) + 1) d\rho\right) \left(\int_{T_0}^{s} g(r) d\rho + \varepsilon\right)^{\frac{1}{2}} ds,
\end{aligned}$$

where $h(t) = \max\left(\frac{h_1(t)}{2}, \frac{h_2(t)}{2}\right)$ a.e. $t \in [T_0, T]$.

2. Main result

A solution to problem (1.1) is understood as follows. A couple $(u, x) : I \to H \times H$ is a solution to (1.1) if and only if u is BVC and x is absolutely continuous satisfying (1.1). In the sequel, we just say a measurable map. The reader will easily identify which type it is (Lebesgue or Borel) from the context.

Let us prove our existence result regarding (1.1).

Theorem 2.1. *Assume that for any* $t \in I$, $B_1(t) : \mathrm{D}(B_1(t)) \subset H \to 2^H$ *is a maximal monotone operator satisfying the following conditions:*

(H_1) *There exists a function* $\rho : I \to [0, +\infty[$ *that is continuous on* I *and nondecreasing with* $\rho(T) < +\infty$, $\rho(0) = 0$ *such that*

$$\operatorname{dis}(B_1(t), B_1(s)) \leq |\rho(t) - \rho(s)|, \quad \textit{for all } t, s \in I.$$

(H_2) *There exists a nonnegative real constant* c *such that*

$$\|B_1^0(t)w\| \leq c(1 + \|w\|) \text{ for } t \in I,\ w \in \mathrm{D}(B_1(t)).$$

(H_3) *The set* $\mathrm{D}(B_1(t))$ *is relatively ball-compact for each* $t \in I$.

Assume that for any $t \in I$, $B_2(t) : \mathrm{D}(B_2(t)) \subset H \to 2^H$ *is a maximal monotone operator satisfying the following conditions:*

(H_1') *There exists a function* $\alpha \in W^{1,2}(I, \mathbb{R})$ *that is nonnegative on* $[0, T[$ *and nondecreasing with* $\alpha(T) < +\infty$, $\alpha(0) = 0$, *such that*

$$\operatorname{dis}(B_2(t), B_2(s)) \leq |\alpha(t) - \alpha(s)|, \quad \textit{for all } t, s \in I.$$

(H_2') *There exists a nonnegative real constant* d *such that*

$$\|B_2^0(t)w\| \leq d(1 + \|w\|) \text{ for } t \in I,\ w \in \mathrm{D}(B_2(t)).$$

(H_3') *The set* $\mathrm{D}(B_2(t))$ *is relatively ball-compact for each* $t \in I$.

Let $f_1 : I \times I \times H \times H \to H$ *be a map such that*

(i) *the map* $f_1(\cdot, \cdot, u, v)$ *is measurable on* $I \times I$ *for each* $(u, v) \in H \times H$ *and the map* $f_1(t, s, \cdot, \cdot)$ *is continuous for each* $(t, s) \in I \times I$,

(ii) *there exists a nonnegative real constant* m *such that*

$$\|f_1(t, s, u, v)\| \leq m(1 + \|u\| + \|v\|) \text{ for all } (t, s, u, v) \in I \times I \times H \times H. \tag{2.1}$$

Let $f_2 : I \times H \times H \to H$ *be a map such that*

(j) *the map* $f_2(\cdot, u, v)$ *is measurable on* I *for each* $(u, v) \in H \times H$ *and the map* $f_2(t, \cdot, \cdot)$ *is continuous for each* $t \in I$,

(jj) *there exists a nonnegative real constant* l *such that*

$$\|f_2(t, u, v)\| \leq l(1 + \|u\| + \|v\|) \text{ for all } (t, u, v) \in I \times H \times H. \tag{2.2}$$

Then, there exist a solution $(u, x) : I \to H \times H$ *to the first-order mixed partially BVC differential system associated to* ρ, *namely* (1.1) *and nonnegative real constants* ξ *and* κ *depending on* c, d, m, l, $\alpha(T)$, $\rho(T)$, $\|x_0\|$, $\|u_0\|$. *Moreover, one has*

$$\left\|\frac{du}{d\rho}(t)\right\| \leq \kappa, \ \|\dot{x}(t)\| \leq \xi(1 + \dot{\alpha}(t)), \quad \text{for any } t \in I. \tag{2.3}$$

Proof. We proceed by a discretization approach.

Part 1: Let us define a subdivision of the interval I by

$$0 = t_0^n < t_1^n < \cdots < t_i^n < t_{i+1}^n < \cdots < t_n^n = T.$$

For every $n \geq 1$ and $i = 0, 1, \ldots, n-1$, set

$$\Delta_{i+1}^n = t_{i+1}^n - t_i^n, \quad \alpha_{i+1}^n = \alpha(t_{i+1}^n) - \alpha(t_i^n), \quad \rho_{i+1}^n = \rho(t_{i+1}^n) - \rho(t_i^n), \tag{2.4}$$

and suppose that

$$\Delta_i^n \leq \Delta_{i+1}^n, \quad \alpha_i^n \leq \alpha_{i+1}^n, \quad \rho_i^n \leq \rho_{i+1}^n \leq \varsigma_n \text{ and } \delta_{i+1}^n = \Delta_{i+1}^n + \alpha_{i+1}^n \leq \epsilon_n, \tag{2.5}$$

where $\epsilon_n = \frac{\delta(T)}{n}$, and the map δ is defined by $\delta(t) = t + \alpha(t)$, $t \in I$, $\varsigma_n = \frac{\rho(T)}{n}$. We remark that $\varsigma_n, \epsilon_n \to 0$ as $n \to +\infty$.
Put $x_0^n = x_0$, $u_0^n = u_0$. Set for $i = 0, \ldots, n-1$

$$\begin{cases} u_{i+1}^n = J_{B_1,\rho_{i+1}^n}^n \Big(u_i^n - \int_{t_i^n}^{t_{i+1}^n} g^{n,i}(\tau) d\rho(\tau) \Big), \\ x_{i+1}^n = J_{B_2,\Delta_{i+1}^n}^n \Big(x_i^n - \int_{t_i^n}^{t_{i+1}^n} f_2(\tau, x_i^n, u_i^n) d\tau \Big), \end{cases}$$

where

$$\begin{cases} J_{B_1,\rho_{i+1}^n}^n = J_{\rho_{i+1}^n}^{B_1(t_{i+1}^n)} = \Big(I_H + \rho_{i+1}^n B_1(t_{i+1}^n) \Big)^{-1}, \\ J_{B_2,\Delta_{i+1}^n}^n = J_{\Delta_{i+1}^n}^{B_2(t_{i+1}^n)} = \Big(I_H + \Delta_{i+1}^n B_2(t_{i+1}^n) \Big)^{-1}, \end{cases} \tag{2.6}$$

with $g^{n,0}(\tau) = \int_0^\tau f_1(\tau, s, x_0^n, u_0^n) d\rho(s)$, $\tau \in [0, t_1^n[$ and for $i = 1, \ldots, n-1$, the map $g^{n,i}$ is defined by

$$g^{n,i}(\tau) = \sum_{j=0}^{i-1} \int_{t_j^n}^{t_{j+1}^n} f_1(\tau, s, x_j^n, u_j^n) d\rho(s) + \int_{t_i^n}^{\tau} f_1(\tau, s, x_i^n, u_i^n) d\rho(s), \quad \tau \in [t_i^n, t_{i+1}^n[. \tag{2.7}$$

Then, note from (2.1), (2.4), and (2.7) that for each $\tau \in [t_i^n, t_{i+1}^n[$

$$\|g^{n,i}(\tau)\| \leq \sum_{j=0}^{i} m\rho_{j+1}^n (1 + \|x_j^n\| + \|u_j^n\|). \tag{2.8}$$

Combining (1.3) with (2.6), one gets

$$u_{i+1}^n \in \mathrm{D}\,(B_1(t_{i+1}^n)), \quad x_{i+1}^n \in \mathrm{D}\,(B_2(t_{i+1}^n)), \tag{2.9}$$

$$u_i^n - \int_{t_i^n}^{t_{i+1}^n} g^{n,i}(\tau) d\rho(\tau) \in u_{i+1}^n + \rho_{i+1}^n B_1(t_{i+1}^n) u_{i+1}^n, \tag{2.10}$$

$$x_i^n - \int_{t_i^n}^{t_{i+1}^n} f_2(\tau, x_i^n, u_i^n) d\tau \in x_{i+1}^n + \Delta_{i+1}^n B_2(t_{i+1}^n) x_{i+1}^n. \tag{2.11}$$

The two last inclusions may be written as follows:

$$-\frac{u_{i+1}^n - u_i^n}{\rho_{i+1}^n} - \frac{1}{\rho_{i+1}^n} \int_{t_i^n}^{t_{i+1}^n} g^{n,i}(\tau) d\rho(\tau) \in B_1(t_{i+1}^n) u_{i+1}^n,$$

$$-\frac{x_{i+1}^n - x_i^n}{\Delta_{i+1}^n} - \frac{1}{\Delta_{i+1}^n} \int_{t_i^n}^{t_{i+1}^n} f_2(\tau, x_i^n, u_i^n) d\tau \in B_2(t_{i+1}^n) x_{i+1}^n.$$

Now, Lemma 1.10 yields

$$\begin{aligned}\|u_{i+1}^n-u_i^n\| &= \|J^n_{B_1,\rho^n_{i+1}}\Big(u_i^n-\int_{t_i^n}^{t_{i+1}^n} g^{n,i}(\tau)d\rho(\tau)\Big)-u_i^n\| \\ &\le \|J^n_{B_1,\rho^n_{i+1}}\Big(u_i^n-\int_{t_i^n}^{t_{i+1}^n} g^{n,i}(\tau)d\rho(\tau)\Big)-J^n_{B_1,\rho^n_{i+1}}(u_i^n)\| \\ &\quad +\|J^n_{B_1,\rho^n_{i+1}}(u_i^n)-u_i^n\| \\ &\le \int_{t_i^n}^{t_{i+1}^n}\|g^{n,i}(\tau)\|d\rho(\tau)+\rho^n_{i+1}\|B_1^0(t_i^n)u_i^n\|+\operatorname{dis}(B_1(t_{i+1}^n),B_1(t_i^n)) \\ &\quad +\Big(\rho^n_{i+1}(1+\|B_1^0(t_i^n)u_i^n\|)\operatorname{dis}(B_1(t_{i+1}^n),B_1(t_i^n))\Big)^{\frac12}.\end{aligned}$$

Then, using (2.4), (2.8), assumptions (H_1)-(H_2) and the fact that $\sqrt{vw}\le\frac12(v+w)$ for nonnegative real constants v, w, gives

$$\begin{aligned}\|u_{i+1}^n-u_i^n\| &\le \rho^n_{i+1}\sum_{j=0}^{i}\rho^n_{j+1}m(1+\|x_j^n\|+\|u_j^n\|)+\rho^n_{i+1}c(1+\|u_i^n\|) \\ &\quad +\frac{\rho^n_{i+1}}{2}(1+c(1+\|u_i^n\|))+\frac32\rho^n_{i+1} \\ &\le m\rho^n_{i+1}\sum_{j=0}^{i}\rho^n_{j+1}(1+\|x_j^n\|+\|u_j^n\|) \\ &\quad +\frac{3c}{2}\rho^n_{i+1}(1+\|u_i^n\|)+2\rho^n_{i+1}. \qquad (2.12)\end{aligned}$$

In a similar way, from Lemma 1.10, one writes

$$\begin{aligned}\|x_{i+1}^n-x_i^n\| &= \|J^n_{B_2,\Delta^n_{i+1}}\Big(x_i^n-\int_{t_i^n}^{t_{i+1}^n} f_2(\tau,x_i^n,u_i^n)d\tau\Big)-x_i^n\| \\ &\le \|J^n_{B_2,\Delta^n_{i+1}}\Big(x_i^n-\int_{t_i^n}^{t_{i+1}^n} f_2(\tau,x_i^n,u_i^n)d\tau\Big)-J^n_{B_2,\Delta^n_{i+1}}(x_i^n)\| \\ &\quad +\|J^n_{B_2,\Delta^n_{i+1}}(x_i^n)-x_i^n\| \\ &\le \int_{t_i^n}^{t_{i+1}^n}\|f_2(\tau,x_i^n,u_i^n)\|d\tau+\Delta^n_{i+1}\|B_2^0(t_i^n)x_i^n\|+\operatorname{dis}(B_2(t_{i+1}^n),B_2(t_i^n)) \\ &\quad +\Big(\Delta^n_{i+1}(1+\|B_2^0(t_i^n)x_i^n\|)\operatorname{dis}(B_2(t_{i+1}^n),B_2(t_i^n))\Big)^{\frac12}.\end{aligned}$$

Next, using (2.2), (2.4), assumptions (H_1')-(H_2') gives

$$\begin{aligned}\|x_{i+1}^n-x_i^n\| &\le \Delta^n_{i+1}l(1+\|x_i^n\|+\|u_i^n\|)+\Delta^n_{i+1}d(1+\|x_i^n\|)+\alpha^n_{i+1} \\ &\quad +\Big(\Delta^n_{i+1}(1+d(1+\|x_i^n\|))\alpha^n_{i+1}\Big)^{\frac12}.\end{aligned}$$

Then, using the fact that $\sqrt{vw} \leq \frac{1}{2}(v+w)$ for nonnegative real constants v, w, gives

$$\begin{aligned} \|x_{i+1}^n - x_i^n\| &\leq \Delta_{i+1}^n l(1+\|x_i^n\|+\|u_i^n\|) + \Delta_{i+1}^n d(1+\|x_i^n\|) \\ &\quad + \frac{\Delta_{i+1}^n}{2}(1+d(1+\|x_i^n\|)) + \frac{3}{2}\alpha_{i+1}^n \\ &\leq \Delta_{i+1}^n l(\|x_i^n\|+\|u_i^n\|) + \Delta_{i+1}^n \frac{3d}{2}\|x_i^n\| \\ &\quad + \Delta_{i+1}^n (l + \frac{3d}{2} + \frac{1}{2}) + \frac{3}{2}\alpha_{i+1}^n, \end{aligned}$$

along with (2.5), it results

$$\|x_{i+1}^n - x_i^n\| \leq \Delta_{i+1}^n (l+\frac{3d}{2})(\|x_i^n\|+\|u_i^n\|) + \delta_{i+1}^n (l + \frac{3d}{2} + 2). \tag{2.13}$$

On the one hand, one writes from (2.12)

$$\begin{aligned} \|u_{i+1}^n\| &\leq \|u_0^n\| + \sum_{j=0}^{i} \|u_{j+1}^n - u_j^n\| \\ &\leq \|u_0^n\| + \sum_{j=0}^{i} m\rho_{j+1}^n \sum_{k=0}^{j} \rho_{k+1}^n (1+\|x_k^n\|+\|u_k^n\|) \\ &\quad + \frac{3c}{2}\sum_{j=0}^{i} \rho_{j+1}^n (1+\|u_j^n\|) + 2\sum_{j=0}^{i}\rho_{j+1}^n. \end{aligned}$$

Note that for $j \leq i$, one has

$$\sum_{k=0}^{j} \rho_{k+1}^n (1+\|x_k^n\|+\|u_k^n\|) \leq \sum_{k=0}^{i} \rho_{k+1}^n (1+\|x_k^n\|+\|u_k^n\|),$$

it follows

$$\begin{aligned} \|u_{i+1}^n\| &\leq \|u_0^n\| + \sum_{j=0}^{i} m\rho_{j+1}^n \sum_{k=0}^{i} \rho_{k+1}^n (1+\|x_k^n\|+\|u_k^n\|) \\ &\quad + \frac{3c}{2}\sum_{k=0}^{i} \rho_{k+1}^n (1+\|u_k^n\|) + 2\sum_{k=0}^{i}\rho_{k+1}^n \\ &\leq \|u_0^n\| + m\rho(T) \sum_{k=0}^{i} \rho_{k+1}^n (1+\|x_k^n\|+\|u_k^n\|) \\ &\quad + \frac{3c}{2}\sum_{k=0}^{i} \rho_{k+1}^n (1+\|u_k^n\|) + 2\sum_{k=0}^{i}\rho_{k+1}^n, \end{aligned}$$

along with (2.4) and (2.5), one obtains

$$\begin{aligned} \|u_{i+1}^n\| &\leq \|u_0\| + (m\rho(T) + \frac{3c}{2})\varsigma_n \sum_{k=0}^{i}(\|x_k^n\| + \|u_k^n\|) + (m\rho(T) + \frac{3c}{2} + 2)\rho(T) \\ &\leq \|u_0\| + m_1\varsigma_n \sum_{k=0}^{i}(\|x_k^n\| + \|u_k^n\|) + m_2\rho(T), \end{aligned} \tag{2.14}$$

where $m_1 = m\rho(T) + \frac{3c}{2}$ and $m_2 = m\rho(T) + \frac{3c}{2} + 2$.

On the other hand, from (2.13), one writes

$$\begin{aligned} \|x_{i+1}^n\| &\leq \|x_0\| + \sum_{j=0}^{i} \|x_{j+1}^n - x_j^n\| \\ &\leq \|x_0\| + l_1 \sum_{j=0}^{i} \Delta_{j+1}^n(\|x_j^n\| + \|u_j^n\|) + l_2 \sum_{j=0}^{i} \delta_{j+1}^n, \end{aligned}$$

along with (2.4) and (2.5), it results

$$\|x_{i+1}^n\| \leq \|x_0\| + l_1\epsilon_n \sum_{k=0}^{i}(\|x_k^n\| + \|u_k^n\|) + l_2\delta(T), \tag{2.15}$$

where $l_1 = l + \frac{3d}{2}$ and $l_2 = l + \frac{3d}{2} + 2$.

Summing (2.14) and (2.15) member to member, one obtains

$$\|u_{i+1}^n\| + \|x_{i+1}^n\| \leq \|u_0\| + \|x_0\| + l_2\delta(T) + m_2\rho(T) + (l_1\epsilon_n + m_1\varsigma_n) \sum_{k=0}^{i}(\|x_k^n\| + \|u_k^n\|).$$

An application of Lemma 1.12 gives

$$\|u_{i+1}^n\| + \|x_{i+1}^n\| \leq \Big(\|u_0\| + \|x_0\| + l_2\delta(T) + m_2\rho(T)\Big) \exp\Big(\sum_{k=0}^{i}(l_1\epsilon_n + m_1\varsigma_n)\Big).$$

Thus, using (2.5), one gets

$$\|u_i^n\| + \|x_i^n\| \leq k, \tag{2.16}$$

where $k = \Big(\|u_0\| + \|x_0\| + l_2\delta(T) + m_2\rho(T)\Big) \exp(l_1\delta(T) + m_1\rho(T))$.

Combining (2.12) and (2.16), it follows

$$\|u_{i+1}^n - u_i^n\| \leq k_1\rho_{i+1}^n, \quad \|u_i^n\| \leq k_1, \tag{2.17}$$

where $k_1 = \max\{k, m_1(1+k) + 2\}$.

Now, coming back to (2.13) with the help of (2.16)

$$\|x_{i+1}^n - x_i^n\| \leq \xi\delta_{i+1}^n, \quad \|x_i^n\| \leq \xi, \tag{2.18}$$

where $\xi = \max\{k, l_1 k + l_2\}$.
For any $n \geq 1$, define the sequences $u_n, x_n : I \to H$ by

$$u_n(t) = u_i^n + \frac{\rho(t) - \rho(t_i^n)}{\rho_{i+1}^n}\left(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} g^{n,i}(\tau)d\rho(\tau)\right) - \int_{t_i^n}^{t} g^{n,i}(\tau)d\rho(\tau), \tag{2.19}$$

$$x_n(t) = x_i^n + \frac{t - t_i^n}{\Delta_{i+1}^n}\left(x_{i+1}^n - x_i^n + \int_{t_i^n}^{t_{i+1}^n} f_2(\tau, x_i^n, u_i^n)d\tau\right) - \int_{t_i^n}^{t} f_2(\tau, x_i^n, u_i^n)d\tau, \tag{2.20}$$

with $u_n(t_{i+1}^n) = u_{i+1}^n$ and $x_n(t_{i+1}^n) = x_{i+1}^n$. Then, the functions $u_n, x_n : I \to H$ are continuous.

Now, by (2.17), we have

$$\sup_n var(u_n(\cdot)) = \sup_n \left(\sum_{i=0}^{n-1} \|u_{i+1}^n - u_i^n\|\right) \leq k_1 \sum_{i=0}^{n-1} \rho_{i+1}^n \leq k_1 \rho(T),$$

and by (2.18), we have

$$\sup_n var(x_n(\cdot)) = \sup_n \left(\sum_{i=0}^{n-1} \|x_{i+1}^n - x_i^n\|\right) \leq \xi \sum_{i=0}^{n-1} \delta_{i+1}^n \leq \xi \delta(T),$$

that is, u_n, x_n are BV maps for any $n \geq 1$.
From (2.19) and (2.20), for $t \in [t_i^n, t_{i+1}^n[$, $i = 0, 1, \ldots, n-1$ and $x_n(T) = x_n^n$, $u_n(T) = u_n^n$, one obtains

$$\frac{du_n}{d\rho}(t) = \frac{1}{\rho_{i+1}^n}\left(u_{i+1}^n - u_i^n + \int_{t_i^n}^{t_{i+1}^n} g^{n,i}(\tau)d\rho(\tau)\right) - g^{n,i}(t), \tag{2.21}$$

$$\dot{x}_n(t) = \frac{1}{\Delta_{i+1}^n}\left(x_{i+1}^n - x_i^n + \int_{t_i^n}^{t_{i+1}^n} f_2(\tau, x_i^n, u_i^n)d\tau\right) - f_2(t, x_i^n, u_i^n). \tag{2.22}$$

Then, (2.10) and (2.11) take the form

$$\begin{cases} -\dfrac{du_n}{d\rho}(t) \in B_1(t_{i+1}^n)u_n(t_{i+1}^n) + g^{n,i}(t) \quad d\rho - \text{a.e. } t \in I, \\ -\dot{x}_n(t) \in B_2(t_{i+1}^n)x_n(t_{i+1}^n) + f_2(t, x_i^n, u_i^n) \quad \text{a.e. } t \in I. \end{cases} \tag{2.23}$$

Using (2.8) and (2.16), one writes

$$\|g^{n,i}(t)\| \leq m\rho(T)(1 + k), \quad \text{for each } t \in I. \tag{2.24}$$

In view of (2.5), (2.17), (2.19), and (2.24), one has

$$\begin{aligned} \|u_n(t) - u_i^n\| &\leq \|u_{i+1}^n - u_i^n\| + 2m\rho(T)(1 + k)\rho_{i+1}^n \\ &\leq \rho_{i+1}^n(k_1 + 2m\rho(T)(1 + k)) \\ &\leq \varsigma_n(k_1 + 2m\rho(T)(1 + k)). \end{aligned} \tag{2.25}$$

Next, observe by (2.2) and (2.16) that

$$\|f_2(t, x_i^n, u_i^n)\| \leq l(1 + k), \quad \text{for each } t \in I. \tag{2.26}$$

Then, combining (2.5), (2.18), (2.20), and (2.26), it follows

$$\begin{aligned}\|x_n(t)-x_i^n\| &\le \|x_{i+1}^n - x_i^n\| + 2l(1+k)\Delta_{i+1}^n\\ &\le \delta_{i+1}^n(\xi+2l(1+k))\\ &\le \epsilon_n(\xi+2l(1+k)).\end{aligned} \tag{2.27}$$

Fix $t_2 \in [t_i^n, t_{i+1}^n[$ and $t_1 \in [t_j^n, t_{j+1}^n[$ with $i<j$. Then, by (2.5), (2.17), and (2.25), putting $k_3 = k_1 + 2m\rho(T)(1+k)$, one simplifies

$$\begin{aligned}\|u_n(t_1)-u_n(t_2)\| &\le \|u_n(t_1)-u_j^n\| + \|u_j^n - u_i^n\| + \|u_i^n - u_n(t_2)\|\\ &\le \|u_j^n - u_i^n\| + 2k_3\varsigma_n\\ &\le \sum_{p=0}^{j-i-1}\|u_{i+p+1}^n - u_{i+p}^n\| + 2k_3\varsigma_n\\ &\le k_1\sum_{p=0}^{j-i-1}\rho_{i+p+1}^n + 2k_3\varsigma_n\\ &= k_1\Big(\rho(t_j^n)-\rho(t_i^n)\Big) + 2k_3\varsigma_n\\ &\le k_1\Big(\rho(t_1)-\rho(t_i^n)\Big) + 2k_3\varsigma_n\\ &= k_1\Big(\rho(t_1)-\rho(t_2)+\rho(t_2)-\rho(t_i^n)\Big) + 2k_3\varsigma_n\\ &\le k_1\Big(\rho(t_1)-\rho(t_2)+\rho(t_{i+1}^n)-\rho(t_i^n)\Big) + 2k_3\varsigma_n\\ &\le k_1\Big(\rho(t_1)-\rho(t_2)\Big) + k_1\rho_{i+1}^n + 2k_3\varsigma_n,\\ &\le k_1\Big(\rho(t_1)-\rho(t_2)\Big) + (k_1+2k_3)\varsigma_n.\end{aligned}$$

Proceeding similarly, using (2.5), (2.18), and (2.27), one gets for any $n \ge 1$ and $0 \le t_2 \le t_1 \le T$

$$\|x_n(t_1)-x_n(t_2)\| \le \xi\Big(\delta(t_1)-\delta(t_2)\Big) + [\xi + 2(\xi+2l(1+k))]\epsilon_n. \tag{2.28}$$

Combining (2.17), (2.21), and (2.24), it follows for $t \in I$,

$$\begin{aligned}\left\|\frac{du_n}{d\rho}(t)\right\| &\le \frac{1}{\rho_{i+1}^n}\|u_{i+1}^n - u_i^n\| + 2m\rho(T)(1+k)\\ &\le k_1 + 2m\rho(T)(1+k) = \kappa.\end{aligned} \tag{2.29}$$

Combining (2.5), (2.18), (2.22), and (2.26), it results for $t \in I$

$$\begin{aligned} \|\dot{x}_n(t)\| &\leq \frac{1}{\Delta^n_{i+1}}\|x^n_{i+1} - x^n_i\| + 2l(1+k) \\ &\leq \xi \frac{\delta^n_{i+1}}{\Delta^n_{i+1}} + 2l(1+k) \\ &\leq \xi\left(1 + \frac{\alpha(t^n_{i+1}) - \alpha(t^n_i)}{t^n_{i+1} - t^n_i}\right) + 2l(1+k). \end{aligned} \tag{2.30}$$

By the absolute continuity of $\alpha(\cdot)$, one has for a.e. $t \in]t^n_i, t^n_{i+1}[$, $\dot{\alpha}(t) = \lim\limits_{n\to\infty} \frac{\alpha(t^n_{i+1})-\alpha(t^n_i)}{t^n_{i+1}-t^n_i}$. Then, there is a Lebesgue measure null-set $Y \subset I$ such that for every $t \in I \setminus Y$, there exists $\sigma_t < +\infty$ such that

$$\|\dot{x}_n(t)\| \leq \sigma_t. \tag{2.31}$$

Observe by (2.18) that

$$\|x^n_{i+1} - x^n_i\| \leq \int_{t^n_i}^{t^n_{i+1}} \theta(s)ds,$$

where the map θ is defined by $\theta(t) = \xi(1 + \dot{\alpha}(t))$ for any $t \in I$.
Next, using the Cauchy–Schwarz inequality, one writes

$$\|x^n_{i+1} - x^n_i\| \leq (t^n_{i+1} - t^n_i)^{1/2}\left(\int_{t^n_i}^{t^n_{i+1}} \theta^2(s)ds\right)^{1/2}.$$

Combining the last inequality with (2.30), noting that $(v+w)^2 \leq 2(v^2+w^2)$ for $v, w \in \mathbb{R}$, one gets

$$\begin{aligned} \|\dot{x}_n\|^2_{L^2(I,H)} &= \sum_{i=0}^{n-1}\int_{t^n_i}^{t^n_{i+1}} \|\dot{x}_n(t)\|^2 dt \\ &\leq \sum_{i=0}^{n-1}\int_{t^n_i}^{t^n_{i+1}} \left(\frac{1}{\Delta^n_{i+1}}\|x^n_{i+1} - x^n_i\| + 2l(1+k)\right)^2 dt \\ &\leq 2\sum_{i=0}^{n-1}\int_{t^n_i}^{t^n_{i+1}} \left(\left(\frac{\|x^n_{i+1} - x^n_i\|}{t^n_{i+1} - t^n_i}\right)^2 + 4l^2(1+k)^2\right)dt \\ &\leq 2\sum_{i=0}^{n-1}\left(\frac{\|x^n_{i+1} - x^n_i\|^2}{t^n_{i+1} - t^n_i} + 4l^2(1+k)^2(t^n_{i+1} - t^n_i)\right) \\ &\leq 2\left(\sum_{i=0}^{n-1}\int_{t^n_i}^{t^n_{i+1}} \theta^2(s)ds + 4l^2(1+k)^2 T\right) \\ &= 2\left(\int_0^T \theta^2(s)ds + 4l^2(1+k)^2 T\right) < +\infty. \end{aligned}$$

Hence, one deduces that

$$\|\dot{x}_n\|_{L^2(I,H)} \leq \zeta = \left[2\left(\int_0^T \theta^2(s)ds + 4l^2(1+k)^2 T\right)\right]^{\frac{1}{2}} < +\infty. \tag{2.32}$$

Part 2: Put for any $n \geq 1$

$$\varphi_n(t) = \begin{cases} 0 & \text{if } t = 0, \\ t_i^n & \text{if } t \in]t_i^n, t_{i+1}^n] \quad \text{for some } i \in \{0, 1, \dots, n-1\}, \end{cases}$$

$$\phi_n(t) = \begin{cases} 0 & \text{if } t = 0, \\ t_{i+1}^n & \text{if } t \in]t_i^n, t_{i+1}^n] \quad \text{for some } i \in \{0, 1, \dots, n-1\}. \end{cases}$$

Then, one gets by (2.4)–(2.5)

$$\lim_{n\to\infty} \varphi_n(t) = t \text{ and } \lim_{n\to\infty} \phi_n(t) = t, \tag{2.33}$$

$$\max\Big(|\rho(\varphi_n(t)) - \rho(t)|, |\rho(\phi_n(t)) - \rho(t)|\Big) \leq \varsigma_n \to 0 \text{ as } n \to \infty. \tag{2.34}$$

Set for all $t \in I$

$$h_{1,n}(t) = \int_0^t f_1(t, s, x_n(\varphi_n(s)), u_n(\varphi_n(s)))d\rho(s),$$

and

$$h_{2,n}(t) = f_2(t, x_n(\varphi_n(t)), u_n(\varphi_n(t))).$$

Hence, from (2.9) and (2.23) for each $n \in \mathbb{N}^*$, one writes

$$-\frac{du_n}{d\rho}(t) \in B_1(\phi_n(t))u_n(\phi_n(t)) + h_{1,n}(t)\; d\rho - \text{a.e } t \in I, \tag{2.35}$$

$$-\dot{x}_n(t) \in B_2(\phi_n(t))x_n(\phi_n(t)) + h_{2,n}(t) \quad \text{a.e } \; t \in I, \tag{2.36}$$

$$u_n(\phi_n(t)) \in \mathrm{D}\,(B_1(\phi_n(t))), \quad x_n(\phi_n(t)) \in \mathrm{D}\,(B_2(\phi_n(t))). \tag{2.37}$$

Recall that x_n is absolutely continuous and by (2.32) for any $t_1, t_2 \in I$, $t_1 \leq t_2$

$$\|x_n(t_2) - x_n(t_1)\| = \left\| \int_{t_1}^{t_2} \dot{x}_n(s)ds \right\| \leq \int_{t_1}^{t_2} \|\dot{x}_n(s)\|ds \leq (t_2 - t_1)^{\frac{1}{2}}\zeta, \tag{2.38}$$

that is, $\{x_n(\cdot) : n \in \mathbb{N}^*\}$ is equicontinuous.
By (2.18), one has $(x_n(\phi_n(t))) \subset \xi\overline{B}_H$, for all $t \in I$. Along with (2.37) and the ball-compactness property in (H_3') entails that the set $\{x_n(\phi_n(t)) : n \in \mathbb{N}^*\}$ is relatively compact in H, for each $t \in I$.
In view of (2.33) and (2.38), one deduces that for all $t \in I$

$$\|x_n(\varphi_n(t)) - x_n(t)\| \to 0 \text{ and } \|x_n(\phi_n(t)) - x_n(t)\| \to 0 \text{ as } n \to \infty.$$

Thus, the set $\{x_n(t) : n \in \mathbb{N}^*\}$ is relatively compact in H, for each $t \in I$. Applying Ascoli's theorem, we can extract a subsequence of $(x_n(\cdot))_n$ that uniformly converges on I to some map $x(\cdot) \in \mathcal{C}(I, H)$ and satisfying $x(0) = x_0$. Hence,

$$\|x_n(\varphi_n(t)) - x(t)\| \to 0 \text{ and } \|x_n(\phi_n(t)) - x(t)\| \to 0 \text{ as } n \to \infty. \tag{2.39}$$

Observe that the sequence $(\dot{x}_n)$ is bounded in $L^2(I, H)$ (see (2.32)). So, one easily deduces from above that

$$(\dot{x}_n) \text{ weakly converges to } \dot{x} \text{ in } L^2(I, H). \tag{2.40}$$

Now, remark by (2.29) that for any $t_1, t_2 \in I$, $t_1 \leq t_2$

$$\|u_n(t_2) - u_n(t_1)\| = \left\| \int_{]t_1,t_2]} du_n \right\| = \left\| \int_{]t_1,t_2]} \frac{du_n}{d\rho}(s) d\rho(s) \right\| \\ \leq \int_{]t_1,t_2]} \left\| \frac{du_n}{d\rho}(s) \right\| d\rho(s) \leq \kappa d\rho(]t_1, t_2]).$$

Then

$$\|u_n(t_2) - u_n(t_1)\| \leq \kappa \Big(\rho(t_2) - \rho(t_1) \Big), \tag{2.41}$$

that is, $\{u_n(\cdot) : n \in \mathbb{N}^*\}$ is equicontinuous.
By (2.17), one has $(u_n(\phi_n(t))) \subset k_1 \overline{B}_H$, for any $t \in I$. Using then (2.37) and the ball-compactness property in (H_3) entail that the set $\{u_n(\phi_n(t)) : n \in \mathbb{N}^*\}$ is relatively compact in H, for each $t \in I$.
In view of (2.34) and (2.41), one deduces that for any $s \in I$

$$\|u_n(\varphi_n(s)) - u_n(s)\| \to 0 \text{ and } \|u_n(\phi_n(s)) - u_n(s)\| \to 0 \text{ as } n \to \infty.$$

Thus, the set $\{u_n(t) : n \in \mathbb{N}^*\}$ is relatively compact in H, for each $t \in I$. Applying Ascoli's theorem, we can extract a subsequence of $(u_n(\cdot))$ that uniformly converges on I to some map $u(\cdot) \in \mathcal{C}(I, H)$ and satisfying $u(0) = u_0$. Hence,

$$\|u_n(\varphi_n(t)) - u(t)\| \to 0 \text{ and } \|u_n(\phi_n(t)) - u(t)\| \to 0 \text{ as } n \to \infty. \tag{2.42}$$

Observe from (2.29) that the sequence $(\frac{du_n}{d\rho})$ is bounded in $L^2(I, H, d\rho)$. So, one easily deduces from above that

$$\left(\frac{du_n}{d\rho} \right) \text{ weakly converges to } \frac{du}{d\rho} \text{ in } L^2(I, H, d\rho). \tag{2.43}$$

From (2.39), (2.42) and the fact that $f_2(t, \cdot, \cdot)$ is continuous for any $t \in I$

$$\lim_{n\to\infty} \|f_2(t, x_n(\varphi_n(t)), u_n(\varphi_n(t))) - f_2(t, x(t), u(t))\| = 0,$$

then, using (2.26) and the Lebesgue dominated convergence theorem gives

$$\lim_{n\to\infty} \int_0^T \|f_2(t, x_n(\varphi_n(t)), u_n(\varphi_n(t))) - f_2(t, x(t), u(t))\|^2 dt = 0.$$

Then, one deduces

$$(f_2(\cdot, x_n(\varphi_n(\cdot)), u_n(\varphi_n(\cdot)))) \text{ weakly converges in } L^2(I, H) \text{ to } f_2(\cdot, x(\cdot), u(\cdot)). \tag{2.44}$$

From (2.39), (2.42) and the fact that $f_1(t, s, \cdot, \cdot)$ is continuous for any $(t, s) \in I \times I$

$$\lim_{n\to\infty} \|f_1(t, s, x_n(\varphi_n(s)), u_n(\varphi_n(s))) - f_1(t, s, x(s), u(s))\| = 0,$$

then, using (2.1) and the fact that the sequences (u_n) and (x_n) are bounded (see (2.16)), the Lebesgue dominated convergence theorem yields

$$\lim_{n\to\infty} \int_0^t f_1(t, s, x_n(\varphi_n(s)), u_n(\varphi_n(s))) d\rho(s) = \lim_{n\to\infty} \int_0^t f_1(t, s, x(s), u(s)) d\rho(s),$$

that is,

$$h_{1,n}(t) \to h(t) \quad as\ n \to \infty,$$

where h is the map defined by

$$h(t) = \int_0^t f_1(t,s,x(s),u(s))d\rho(s) \text{ for all } t \in I.$$

Note that $h_{1,n}(t) = g^{n,i}(t)$ for each $t \in [t_i^n, t_{i+1}^n[, i = 0, n-1$. Then, from (2.24), one writes

$$\|h_{1,n}(t)\| \leq m\rho(T)(1+k). \tag{2.45}$$

The Lebesgue dominated convergence theorem then, gives

$$(h_{1,n}(\cdot)) \text{ converges in } L^2(I,H,d\rho) \text{ to } h(\cdot). \tag{2.46}$$

Part 3: We are going to establish that

$$-\frac{du}{d\rho}(t) \in B_1(t)u(t) + \int_0^t f_1(t,s,x(s),u(s))d\rho(s) \quad d\rho - \text{a.e. } t \in I, \tag{2.47}$$

$$-\dot{x}(t) \in B_2(t)x(t) + f_2(t,x(t),u(t)) \quad \text{a.e. } t \in I, \tag{2.48}$$

$$u(t) \in \mathrm{D}\,(B_1(t)),\ x(t) \in \mathrm{D}\,(B_2(t)) \quad t \in I, \tag{2.49}$$

$$u(0) = u_0 \in \mathrm{D}\,(B_1(0)),\ x(0) = x_0 \in \mathrm{D}\,(B_2(0)).$$

We show the first inclusion in (2.49). Recall that $u_n(\phi_n(t)) \in \mathrm{D}\,(B_1(\phi_n(t)))$ for all $t \in I$ (see (2.37)). Combining (H_1) and (2.34) yields

$$\text{dis}\,(B_1(\phi_n(t)), B_1(t)) \leq |\rho(\phi_n(t)) - \rho(t)| \leq \varsigma_n \to 0 \text{ as } n \to \infty. \tag{2.50}$$

Remark in view of (H_2) and (2.17), that the sequence $(B_1^0(\phi_n(t))u_n(\phi_n(t)))$ is bounded in H. Extracting a subsequence, the latter weakly converges to some element in H. As the sequence $(u_n(\phi_n(t)))$ converges to $u(t)$ in H (see (2.42)), by the application of Lemma 1.9, it follows that $u(t) \in \mathrm{D}\,(B_1(t))$, $t \in I$.
Next, we establish the second inclusion in (2.49).
Recall that $x_n(\phi_n(t)) \in \mathrm{D}\,(B_2(\phi_n(t)))$ for all $t \in I$ (see (2.37)). Combining (H_1'), the continuity of $\alpha(\cdot)$ and (2.33) give

$$\text{dis}\,(B_2(\phi_n(t)), B_2(t)) \leq |\alpha(\phi_n(t)) - \alpha(t)| \to 0 \text{ as } n \to \infty. \tag{2.51}$$

Remark in view of (H_2') and (2.18) that the sequence $(B_2^0(\phi_n(t))x_n(\phi_n(t)))$ is bounded in H. Extracting a subsequence, the latter weakly converges to some element in H. As the sequence $(x_n(\phi_n(t)))$ converges to $x(t)$ in H (see (2.39)), by the application of Lemma 1.9, it results that $x(t) \in \mathrm{D}\,(B_2(t))$, $t \in I$.
Now, let us state (2.48). From (2.40) and (2.44), the sequence $(\dot{x}_n + h_{2,n})$ weakly converges in $L^2(I,H)$ to $\dot{x}(\cdot) + f_2(\cdot,x(\cdot),u(\cdot))$. Hence, one finds a sequence (η_j) such that for each $j \in \mathbb{N}$, $\eta_j \in co\{\dot{x}_i + h_{2,i},\ i \geq j\}$ and (η_j) converges to $\dot{x}(\cdot) + f_2(\cdot,x(\cdot),u(\cdot))$ in $L^2(I,H)$. Extracting a subsequence not relabeled (η_j) converges to $\dot{x}(\cdot) + f_2(\cdot,x(\cdot),u(\cdot))$ a.e. More precisely, one finds a subset X of I (its Lebesgue measure equals zero), and a subsequence $(j_p) \subset \mathbb{N}$ satisfying for any $t \in I \setminus X$, $\eta_{j_p}(t) \longrightarrow \dot{x}(t) + f_2(t,x(t),u(t))$. Thus, for $t \in I \setminus X$

$$\dot{x}(t) + f_2(t,x(t),u(t)) \in \bigcap_{p\in\mathbb{N}} \overline{co}\{\dot{x}_i(t) + h_{2,i}(t),\ i \geq j_p\},$$

that is, for any $t \in I \setminus X$ and any $e \in H$

$$\langle \dot{x}(t) + f_2(t,x(t),u(t)), e\rangle \leq \limsup_{n\to\infty} \langle \dot{x}_n(t) + h_{2,n}(t), e\rangle. \tag{2.52}$$

Remember that $x(t) \in \mathrm{D}(B_2(t))$, for each $t \in I$. To show that $-\dot{x}(t) \in B_2(t)x(t)+f_2(t,x(t),u(t))$ a.e. $t \in I$, let us prove

$$\langle \dot{x}(t) + f_2(t,x(t),u(t)), x(t) - w\rangle \leq \langle B_2^0(t)w, w - x(t)\rangle \quad \text{a.e. } t \in I,$$

for each $w \in \mathrm{D}(B_2(t))$, using Lemma 1.8.

Let $w \in \mathrm{D}(B_2(t))$. Let us use Lemma 1.11 with maximal monotone operators $B_2(\phi_n(t))$ and $B_2(t)$ verifying (2.51) with $w_n \in \mathrm{D}(B_2(\phi_n(t)))$

$$w_n \to w \text{ and } B_2^0(\phi_n(t))w_n \to B_2^0(t)w. \tag{2.53}$$

Let (2.36) be satisfied for each $n \geq 1$ on $I \setminus X_n$ (where X_n is a Lebesgue null subset of I). As $B_2(t)$ is monotone for each $t \in I$, then one writes

$$\langle \dot{x}_n(t) + h_{2,n}(t), x_n(\phi_n(t)) - w_n\rangle \leq \langle B_2^0(\phi_n(t))w_n, w_n - x_n(\phi_n(t))\rangle. \tag{2.54}$$

From (2.26), (2.31), and (2.54), one obtains for $t \in I \setminus (\bigcup_{n\in\mathbb{N}} X_n \cup X \cup Y)$,

$$\begin{aligned}
\langle \dot{x}_n(t) + h_{2,n}(t), x(t) - w\rangle &= \langle \dot{x}_n(t) + h_{2,n}(t), x_n(\phi_n(t)) - w_n\rangle \\
&\quad + \langle \dot{x}_n(t) + h_{2,n}(t), (x(t) - x_n(\phi_n(t))) - (w - w_n)\rangle \\
&\leq \langle B_2^0(\phi_n(t))w_n, w_n - x_n(\phi_n(t))\rangle \\
&\quad + (\sigma_t + l(1+k))(\|x_n(\phi_n(t)) - x(t)\| + \|w_n - w\|).
\end{aligned}$$

The convergence modes in (2.39) and (2.53) yield

$$\limsup_{n\to\infty} \langle \dot{x}_n(t) + h_{2,n}(t), x(t) - w\rangle \leq \langle B_2^0(t)w, w - x(t)\rangle.$$

Coming back to (2.52), it results

$$\langle \dot{x}(t) + f_2(t,x(t),u(t)), x(t) - w\rangle \leq \langle B_2^0(t)w, w - x(t)\rangle \quad \text{a.e. } t \in I.$$

The differential inclusion (2.48) is then proved.
Now, let us state (2.47). From (2.43) and (2.46), it results that $(\frac{du_n}{d\rho} + h_{1,n})$ weakly converges in $L^2(I,H,d\rho)$ to $\frac{du}{d\rho}(\cdot) + h(\cdot)$. Hence, one finds a sequence (ζ_j) such that for each $j \in \mathbb{N}$, $\zeta_j \in co\{\frac{du_i}{d\rho} + h_{1,i},\ i \geq j\}$ and (ζ_j) converges to $\frac{du}{d\rho}(\cdot) + h(\cdot)$ in $L^2(I,H,d\rho)$. Extracting a subsequence not relabeled (ζ_j) converges to $\frac{du}{d\rho}(\cdot)+h(\cdot)$ $d\rho$ a.e. More precisely, one finds a subset S of I (its $d\rho$-measure equals zero), and a subsequence $(j_p) \subset \mathbb{N}$ satisfying for any $t \in I \setminus S$, $(\zeta_{j_p}(t))$ converges to $\frac{du}{d\rho}(t) + h(t)$. Hence, for $t \in I \setminus S$

$$\frac{du}{d\rho}(t) + h(t) \in \bigcap_{p\in\mathbb{N}} \overline{co}\{\frac{du_i}{d\rho}(t) + h_{1,i}(t),\ i \geq j_p\},$$

that is, for any $t \in I \setminus S$ and any $e \in H$

$$\langle \frac{du}{d\rho}(t) + h(t), e\rangle \leq \limsup_{n\to\infty} \langle \frac{du_n}{d\rho}(t) + h_{1,n}(t), e\rangle. \tag{2.55}$$

Remember that $u(t) \in \mathrm{D}(B_1(t))$, for each $t \in I$. To show that $-\frac{du}{d\rho}(t) \in B_1(t)u(t) + h(t)$ $d\rho$-a.e. $t \in I$, let us prove that

$$\langle \frac{du}{d\rho}(t) + h(t), u(t) - w\rangle \leq \langle B_1^0(t)w, w - u(t)\rangle \quad d\rho - \text{a.e. } t \in I,$$

for each $w \in \mathrm{D}\,(B_1(t))$, using Lemma 1.8.
Let $w \in \mathrm{D}\,(B_1(t))$. Let us use Lemma 1.11 with maximal monotone operators $B_1(\phi_n(t))$ and $B_1(t)$ verifying (2.50) with $w_n \in \mathrm{D}\,(B_1(\phi_n(t)))$

$$w_n \to w \text{ and } B_1^0(\phi_n(t))w_n \to B_1^0(t)w. \tag{2.56}$$

Let (2.35) be satisfied for each $n \geq 1$ on $I \setminus S_n$ (where S_n is a $d\rho$-null subset of I). As $B_1(t)$ is monotone for each $t \in I$, then one writes

$$\langle \frac{du_n}{d\rho}(t) + h_{1,n}(t), u_n(\phi_n(t)) - w_n \rangle \leq \langle B_1^0(\phi_n(t))w_n, w_n - u_n(\phi_n(t)) \rangle. \tag{2.57}$$

From (2.29), (2.45), and (2.57), one gets for $t \in I \setminus (\bigcup_{n\in\mathbb{N}} S_n \cup S)$,

$$\begin{aligned}
\langle \frac{du_n}{d\rho}(t) \;&+\; h_{1,n}(t), u(t) - w \rangle \\
&= \langle \frac{du_n}{d\rho}(t) + h_{1,n}(t), u_n(\phi_n(t)) - w_n \rangle \\
&\quad + \langle \frac{du_n}{d\rho}(t) + h_{1,n}(t), (u(t) - u_n(\phi_n(t))) - (w - w_n) \rangle \\
&\leq \langle B_1^0(\phi_n(t))w_n, w_n - u_n(\phi_n(t)) \rangle \\
&\quad + (\kappa + m\rho(T)(1+k))(\|u_n(\phi_n(t)) - u(t)\| + \|w_n - w\|).
\end{aligned}$$

The convergence modes in (2.42) and (2.56) yield

$$\limsup_{n\to\infty} \langle \frac{du_n}{d\rho}(t) + h_{1,n}(t), u(t) - w \rangle \leq \langle B_1^0(t)w, w - u(t) \rangle.$$

Coming back to (2.55), it follows

$$\langle \frac{du}{d\rho}(t) + h(t), u(t) - w \rangle \leq \langle B_1^0(t)w, w - u(t) \rangle \quad d\rho - \text{a.e. } t \in I.$$

The differential inclusion (2.47) is then proved.
Thus, the problem (1.1) admits a solution $(u, x) : I \to H \times H$.
In view of (2.41), respectively, (2.28), letting $n \to \infty$ yields for $0 \leq t_1 \leq t_2 \leq T$,

$$\begin{aligned}
&\|u(t_2) - u(t_1)\| \leq \kappa(\rho(t_2) - \rho(t_1)), \\
&\|x(t_2) - x(t_1)\| \leq \xi(t_2 - t_1 + \alpha(t_2) - \alpha(t_1)).
\end{aligned}$$

The estimates in (2.3) are satisfied. This ends the proof of our theorem. □

We close this section by adding extra conditions to ensure the uniqueness of the solution of a related dynamical system to (1.1).

Corollary 2.2. *Assume that for any $t \in I$, $B_1(t) : \mathrm{D}\,(B_1(t)) \subset H \to 2^H$ is a maximal monotone operator satisfying (H_2)-(H_3). Suppose moreover that*
(H_1'') there exists a real nonnegative constant $\beta \geq 0$ such that

$$\mathrm{dis}\,(B_1(t), B_1(s)) \leq \beta(t - s) \text{ for } 0 \leq s \leq t \leq T.$$

Assume that for any $t \in I$, $B_2(t) : \mathrm{D}\,(B_2(t)) \subset H \to 2^H$ is a maximal monotone operator satisfying (H_1')-(H_2')-(H_3').
Let $f_1 : I \times I \times H \times H \to H$ be a map such that assumptions (i)-(ii) hold

true. Suppose moreover that for every $\eta > 0$, there exists a nonnegative function $\psi_\eta(\cdot) \in L^2(I, \mathbb{R})$ such that for all $t, s \in I$ and any $u_1, u_2, v_1, v_2 \in \overline{B}_H[0, \eta]$, one has

$$\|f_1(t, s, u_1, v_1) - f_1(t, s, u_2, v_2)\| \le \psi_\eta(t)\left(\|u_1 - u_2\| + \|v_1 - v_2\|\right). \tag{2.58}$$

Let $f_2 : I \times H \times H \to H$ be a map such that assumptions (j)-(jj) hold true. Suppose moreover that for every $\eta > 0$, there exists a nonnegative function $\sigma_\eta(\cdot) \in L^2(I, \mathbb{R})$ such that for all $t \in I$ and any $u_1, u_2, v_1, v_2 \in \overline{B}_H[0, \eta]$, one has

$$\|f_2(t, u_1, v_1) - f_2(t, u_2, v_2)\| \le \sigma_\eta(t)\left(\|u_1 - u_2\| + \|v_1 - v_2\|\right). \tag{2.59}$$

Then, there exists a unique solution $(u, x) : I \to H \times H$ (u is Lipschitz-continuous and x is absolutely continuous) to the following problem:

$$\begin{cases} -\dot{u}(t) \in B_1(t)u(t) + \displaystyle\int_0^t f_1(t, s, x(s), u(s))ds \quad \text{a.e. } t \in I, \\ -\dot{x}(t) \in B_2(t)x(t) + f_2(t, x(t), u(t)) \quad \text{a.e. } t \in I, \\ (u(0), x(0)) = (u_0, x_0) \in \mathrm{D}\left(B_1(0)\right) \times \mathrm{D}\left(B_2(0)\right), \end{cases} \tag{2.60}$$

with

$$\|\dot{u}(t)\| \le \beta\kappa_1, \quad \|\dot{x}(t)\| \le \ \xi(1 + \dot{\alpha}(t)), \text{for any } t \in I, \tag{2.61}$$

for nonnegative real constants ξ and κ_1 depending on c, d, m, l, $\alpha(T)$, T, $\|x_0\|$, $\|u_0\|$.

Proof. **Existence of the solution:** If $\beta \le 1$, then by (H_1'') one gets

$$\operatorname{dis}\left(B_1(t), B_1(s)\right) \le t - s, \quad \text{for} \quad 0 \le s \le t \le T.$$

By considering $\rho(t) = t$ ($d\rho = dt$), then assumption (H_1) holds true. Hence, Theorem 2.1 guarantees a solution to our problem (2.60).
If $\beta \ge 1$, then combining (H_1'') with the property (1.4), one obtains for $0 \le s \le t \le T$

$$\operatorname{dis}\left(\frac{1}{\beta}B_1(t), \frac{1}{\beta}B_1(s)\right) \le \operatorname{dis}\left(B_1(t), B_1(s)\right) \le \beta(t - s).$$

By considering $\rho(t) = \beta t$ ($d\rho(t) = \beta dt$) to the maximal monotone operator $A(t) = \frac{1}{\beta}B_1(t)$ for all $t \in I$, then assumption (H_1) holds true. Define the perturbation h by $h(t, s, x, u) = \frac{1}{\beta^2}f_1(t, s, x, u)$ for all $(t, s, x, u) \in I \times I \times H \times H$, then, Theorem 2.1 guarantees a solution to the problem

$$\begin{cases} -\dfrac{du}{d\rho}(t) \in \dfrac{1}{\beta}B_1(t)u(t) + \dfrac{1}{\beta}\displaystyle\int_0^t \dfrac{1}{\beta}f_1(t, s, x(s), u(s))d\rho(s) \quad d\rho - \text{a.e. } t \in I, \\ -\dot{x}(t) \in B_2(t)x(t) + f_2(t, x(t), u(t)) \quad \text{a.e. } t \in I, \\ (u(0), x(0)) = (u_0, x_0) \in \mathrm{D}\left(B_1(0)\right) \times \mathrm{D}\left(B_2(0)\right). \end{cases}$$

The first differential inclusion in the latter system may be rewritten as follows:

$$-\frac{du}{dt}(t)\frac{dt}{d\rho}(t) \in \frac{1}{\beta}B_1(t)u(t) + \frac{1}{\beta}\int_0^t \frac{1}{\beta}f_1(t, s, x(s), u(s))d\rho(s) \quad d\rho - a.e \;\; t \in I,$$

that is,

$$-\dot{u}(t) \in B_1(t)u(t) + \int_0^t f_1(t, s, x(s), u(s))ds \text{ a.e } \; t \in I.$$

This ensures the associated existence result to our problem (2.60).

Furthermore, coming back to (2.3), one deduces the estimates in (2.61).

Uniqueness of the solution: Let (u_1, x_1), (u_2, x_2) be two solutions to the coupled system (2.60). Since the solution is bounded, then, there exists η such that $\|u_i(t)\| \leq \eta$ and $\|x_i(t)\| \leq \eta$ for all $t \in I$, $i = 1, 2$.
Now, $B_1(t)$ is monotone, for each $t \in I$, ensures that

$$\langle \dot{u}_1(t) - \dot{u}_2(t), u_1(t) - u_2(t)\rangle \leq$$

$$\left\langle \int_0^t f_1(t, s, x_1(s), u_1(s))ds - \int_0^t f_1(t, s, x_2(s), u_2(s))ds, u_2(t) - u_1(t)\right\rangle. \quad (2.62)$$

Moreover, note that $v + w \leq \sqrt{2}(v^2 + w^2)^{\frac{1}{2}}$ for $v, w \geq 0$ along with the Lipschitz behavior of f_1 in (2.58), one writes

$$\begin{aligned}
&\left\langle \int_0^t f_1(t, s, x_1(s), u_1(s))ds - \int_0^t f_1(t, s, x_2(s), u_2(s))ds, u_2(t) - u_1(t)\right\rangle \\
&\quad \leq \left(\int_0^t \|f_1(t, s, x_1(s), u_1(s)) - f_1(t, s, x_2(s), u_2(s))\|ds\right) \|u_1(t) - u_2(t)\| \\
&\quad \leq \psi_\eta(t)\|u_1(t) - u_2(t)\| \int_0^t \Big(\|u_1(s) - u_2(s)\| + \|x_1(s) - x_2(s)\|\Big)ds \\
&\quad \leq \sqrt{2}\psi_\eta(t)\|u_1(t) - u_2(t)\| \int_0^t \Big(\|u_1(s) - u_2(s)\|^2 + \|x_1(s) - x_2(s)\|^2\Big)^{\frac{1}{2}} ds.
\end{aligned}$$

Observe that

$$\|u_1(t) - u_2(t)\| \leq \left(\|u_1(t) - u_2(t)\|^2 + \|x_1(t) - x_2(t)\|^2\right)^{\frac{1}{2}},$$

since $v \leq (v^2 + w^2)^{\frac{1}{2}}$ for $v \geq 0$, $w \in \mathbb{R}$. Then, define the map y by

$$y(t) = \|u_1(t) - u_2(t)\|^2 + \|x_1(t) - x_2(t)\|^2, \quad \text{for any} \quad t \in I. \quad (2.63)$$

Then, it follows

$$\begin{aligned}
&\left\langle \int_0^t f_1(t, s, x_1(s), u_1(s))ds - \int_0^t f_1(t, s, x_2(s), u_2(s))ds, u_2(t) - u_1(t)\right\rangle \\
&\qquad\qquad \leq \sqrt{2}\psi_\eta(t)(y(t))^{\frac{1}{2}} \int_0^t (y(s))^{\frac{1}{2}} ds. \quad (2.64)
\end{aligned}$$

Hence, combining (2.62)-(2.64) yields

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u_1(t) - u_2(t)\|^2 &= \langle u_1(t) - u_2(t), \dot{u}_1(t) - \dot{u}_2(t)\rangle \\
&\leq \sqrt{2}\psi_\eta(t)(y(t))^{\frac{1}{2}} \int_0^t (y(s))^{\frac{1}{2}} ds. \quad (2.65)
\end{aligned}$$

Now, using the fact that $w^2 + wz \leq \frac{3}{2}(w^2 + z^2)$ for each $w, z \in \mathbb{R}_+$ and the Lipschitz behavior of f_2 in (2.59), one finds a nonnegative function $\sigma_\eta(\cdot) \in L^2(I, \mathbb{R})$ such that $f_2(t,\cdot,\cdot)$ is $\sigma_\eta(t)$-Lipschitz on $\overline{B}_H[0,\eta]\times\overline{B}_H[0,\eta]$ for each $t\in I$. Hence

$$\begin{aligned}&\langle f_2(t,x_1(t),u_1(t))-f_2(t,x_2(t),u_2(t)),x_2(t)-x_1(t)\rangle\\ &\qquad\le \sigma_\eta(t)\Big(\|u_1(t)-u_2(t)\|+\|x_1(t)-x_2(t)\|\Big)\|x_1(t)-x_2(t)\|\\ &\qquad= \sigma_\eta(t)\Big(\|x_1(t)-x_2(t)\|^2+\|u_1(t)-u_2(t)\|\|x_1(t)-x_2(t)\|\Big)\\ &\qquad\le \frac{3\sigma_\eta(t)}{2}\Big(\|u_1(t)-u_2(t)\|^2+\|x_1(t)-x_2(t)\|^2\Big).\end{aligned}\tag{2.66}$$

As $B_2(t)$ is monotone for each $t\in I$, one writes

$$\langle \dot{x}_1(t)-\dot{x}_2(t),x_1(t)-x_2(t)\rangle\le\langle f_2(t,x_1(t),u_1(t))-f_2(t,x_2(t),u_2(t)),x_2(t)-x_1(t)\rangle.\tag{2.67}$$

Hence, combining (2.63), (2.66)-(2.67) yields

$$\frac{1}{2}\frac{d}{dt}\|x_1(t)-x_2(t)\|^2=\langle x_1(t)-x_2(t),\dot{x}_1(t)-\dot{x}_2(t)\rangle\le\frac{3\sigma_\eta(t)}{2}y(t).\tag{2.68}$$

From (2.65) and (2.68), one obtains

$$\dot{y}(t)\le 3\sigma_\eta(t)y(t)+2\sqrt{2}\psi_\eta(t)(y(t))^{\frac{1}{2}}\int_0^t(y(s))^{\frac{1}{2}}ds.$$

Recall that $\|u_1(0)-u_2(0)\|=0$ and $\|x_1(0)-x_2(0)\|=0$ and by assumption both $\psi_\eta(\cdot)$ and $\sigma_\eta(\cdot)$ are nonnegative $L^2(I,\mathbb{R})$-functions. Thanks to Lemma 1.13, it results that $(u_1,x_1)=(u_2,x_2)$ and the solution is unique. □

3. A problem in control theory

In the reminder of this section, let $\mathcal{Z}_1,\mathcal{Z}_2$ be convex compact in $\mathcal{C}(I,H)$. Denote by $\mathcal{Y}_1,\mathcal{Y}_2$ the sets defined by

$$\mathcal{Y}_1=\{y_1\in\mathcal{C}(I,H),\quad y_1(t)=y_0^1+\int_0^t\dot{y}_1(s)ds\quad\text{for all } t\in I,\quad \dot{y}_1(t)\in\Gamma_1\},$$

and

$$\mathcal{Y}_2=\{y_2\in\mathcal{C}(I,H),\quad y_2(t)=y_0^2+\int_0^t\dot{y}_2(s)ds\quad\text{for all } t\in I,\quad \dot{y}_2(t)\in\Gamma_2\},$$

where Γ_1, Γ_2 are convex compact subsets of H.
It is clear that the sets $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are convex compact in $\mathcal{C}(I,H)$.

Let us begin this section by establishing the existence result regarding problem (1.2).

Theorem 3.1. *Let for $(t,x)\in I\times H$, $B_1(t,x):\mathrm{D}(B_1(t,x))\subset H\to 2^H$ be a maximal monotone operator such that*

$(H^1_{B_1})$ *there exist λ_1, $\beta\ge 0$ such that*

$$\operatorname{dis}(B_1(t,v),B_1(s,w))\le\beta(t-s)+\lambda_1\|v-w\|,\quad \textit{for all } t,s\in I(s\le t),\ \textit{for all } v,w\in H;$$

$(H^2_{B_1})$ *there exists a nonnegative real constant c_1 such that*

$$\|B_1^0(t,v)w\|\le c_1(1+\|v\|+\|w\|)\quad\text{for}\quad t\in I,\ v\in H,\ w\in\mathrm{D}(B_1(t,v));$$

($H^3_{B_1}$) *for any bounded subset $X \subset H$, the set $\mathrm{D}\,(B_1(I \times X))$ is relatively ball-compact.*

Let for $(t,x) \in I \times H$, $B_2(t,x) : \mathrm{D}\,(B_2(t,x)) \subset H \to 2^H$ be a maximal monotone operator such that

($H^1_{B_2}$) *there exist $\lambda_2 \geq 0$, and a function $\alpha \in W^{1,2}(I,\mathbb{R})$, which is nonnegative on $[0,T[$ and nondecreasing with $\alpha(T) < +\infty$ and $\alpha(0) = 0$ such that*

$$\mathrm{dis}\,(B_2(t,v), B_2(s,w)) \leq |\alpha(t)-\alpha(s)|+\lambda_2\|v-w\|, \quad \textit{for all } t,s \in I, \quad \textit{for all } v,w \in H;$$

($H^2_{B_2}$) *there exists a nonnegative real constant d_1 such that*

$$\|B_2^0(t,v)w\| \leq d_1(1+\|v\|+\|w\|) \text{ for } t \in I,\ v \in H,\ w \in \mathrm{D}\,(B_2(t,v));$$

($H^3_{B_2}$) *for any bounded subset $X \subset H$, the set $\mathrm{D}\,(B_2(I \times X))$ is relatively ball-compact.*

Let $f_1 : I \times I \times H \times H \times H \to H$ be a map such that

($h^1_{f_1}$) *the map $f_1(\cdot,\cdot,u,v,w)$ is measurable on $I \times I$ for each $(u,v,w) \in H \times H \times H$, and $f_1(t,s,\cdot,\cdot,\cdot)$ is continuous for each $(t,s) \in I \times I$;*

($h^2_{f_1}$) *there exists $m_1 \geq 0$ such that*

$$\|f_1(t,s,u,v,w)\| \leq m_1(1+\|u\|+\|v\|+\|w\|) \text{ for all } (t,s,u,v,w) \in I\times I\times H\times H\times H; \tag{3.1}$$

($h^3_{f_1}$) *for every $\eta > 0$, there exists a nonnegative function $\psi_\eta(\cdot) \in L^2(I,\mathbb{R})$ such that for all $t,s \in I$ and any $u_1,u_2,v_1,v_2 \in \overline{B}_H[0,\eta]$ and $w \in H$, one has*

$$\|f_1(t,s,u_1,v_1,w) - f_1(t,s,u_2,v_2,w)\| \leq \psi_\eta(t)\left(\|u_1-u_2\|+\|v_1-v_2\|\right). \tag{3.2}$$

Let $f_2 : I \times H \times H \times H \to H$ be a map such that

($h^1_{f_2}$) *the map $f_2(\cdot,u,v,w)$ is measurable on I for each $(u,v,w) \in H \times H \times H$, and $f_2(t,\cdot,\cdot,\cdot)$ is continuous for each $t \in I$;*

($h^2_{f_2}$) *there exists $l_1 \geq 0$ such that*

$$\|f_2(t,u,v,w)\| \leq l_1(1+\|u\|+\|v\|+\|w\|) \text{ for all } (t,u,v,w) \in I\times H\times H\times H; \tag{3.3}$$

($h^3_{f_2}$) *for every $\eta > 0$, there exists a nonnegative function $\sigma_\eta(\cdot) \in L^2(I,\mathbb{R})$ such that for all $t \in I$ and any $u_1,u_2,v_1,v_2 \in \overline{B}_H[0,\eta]$ and $w \in H$, one has*

$$\|f_2(t,u_1,v_1,w) - f_2(t,u_2,v_2,w)\| \leq \sigma_\eta(t)\left(\|u_1-u_2\|+\|v_1-v_2\|\right). \tag{3.4}$$

Then, there exists a unique solution $(u,x) : I \to H \times H$ to the first-order system (1.2), for any $(y_1,y_2) \in \mathcal{Y}_1\times\mathcal{Y}_2$, $(z_1,z_2) \in \mathcal{Z}_1\times\mathcal{Z}_2$, and nonnegative real constants ξ_2 and κ_2 depending on c_2, d_2, m_1, l_1, τ_1, τ_2, T, $\gamma_1(T)$, $\gamma_2(T)$, $\|x_0\|$, $\|u_0\|$, where the maps γ_1 and γ_2 are defined, for each $t \in I$, by

$$\gamma_1(t) = (\beta+\lambda_1\beta_1)t, \quad \gamma_2(t) = \int_0^t (\dot\alpha(s)+\lambda_2\|\dot y_2(s)\|)ds,$$

and

$$c_2 = c_1(1+\|y_0^1\|+\int_0^T \|\dot y_1(s)\|ds), \quad d_2 = d_1(1+\|y_0^2\|+\int_0^T \|\dot y_2(s)\|ds).$$

Moreover, one has

$$\|\dot{u}(t)\| \leq \kappa_2, \quad \|\dot{x}(t)\| \leq \xi_2(1+\dot{\gamma_2}(t)), \quad \text{for } t \in I, \tag{3.5}$$

(*where u is Lipschitz-continuous and x is absolutely continuous*).

Proof. Fix $(y_1, y_2) \in \mathcal{Y}_1 \times \mathcal{Y}_2$. Then, define the maximal monotone operators $A_1^{y_1}$, $A_2^{y_2}$ by

$$A_1^{y_1}(t) = B_1(t, y_1(t)), \quad A_2^{y_2}(t) = B_2(t, y_2(t)), \quad \text{for each } t \in I.$$

Let $s, t \in I$ such that $0 \leq s \leq t \leq T$, then one has from $(H^1_{B_1})$

$$\begin{aligned}\operatorname{dis}(A_1^{y_1}(t), A_1^{y_1}(s)) &\leq \beta(t-s) + \lambda_1\|y_1(t) - y_1(s)\| \\ &\leq \int_s^t (\beta + \lambda_1\|\dot{y}_1(\tau)\|)d\tau.\end{aligned}$$

As $y_1 \in \mathcal{Y}_1$, then, there is a nonnegative real constant β_1 such that $\|\dot{y}_1(\tau)\| \leq \beta_1$, for each $\tau \in I$. Then,

$$\operatorname{dis}(A_1^{y_1}(t), A_1^{y_1}(s)) \leq \gamma_1(t) - \gamma_1(s),$$

where $\gamma_1(\cdot)$ is defined by

$$\gamma_1(t) = (\beta + \lambda_1\beta_1)t, \ \text{ for all } t \in I.$$

Similarly, using $(H^1_{B_2})$, one has

$$\begin{aligned}\operatorname{dis}(A_2^{y_2}(t), A_2^{y_2}(s)) &\leq |\alpha(t) - \alpha(s)| + \lambda_2\|y_2(t) - y_2(s)\| \\ &\leq \int_s^t (\dot{\alpha}(\tau) + \lambda_2\|\dot{y}_2(\tau)\|)d\tau = \gamma_2(t) - \gamma_2(s),\end{aligned}$$

where $\gamma_2(\cdot) \in W^{1,2}(I, \mathbb{R})$ is defined by

$$\gamma_2(t) = \int_0^t (\dot{\alpha}(s) + \lambda_2\|\dot{y}_2(s)\|)ds, \ \text{ for all } t \in I.$$

Moreover, by $(H^2_{B_1})$, one has

$$\begin{aligned}\|(A_1^{y_1})^0(t)w\| = \|B_1^0(t, y_1(t))w\| &\leq c_1(1 + \|y_1(t)\| + \|w\|) \\ &\leq c_2(1 + \|w\|),\end{aligned}$$

where $c_2 = c_1(1 + \|y_0^1\| + \int_0^T \|\dot{y}_1(s)\|ds)$.
Similarly, by $(H^2_{B_2})$, one has

$$\begin{aligned}\|(A_2^{y_2})^0(t)w\| = \|B_2^0(t, y_2(t))w\| &\leq d_1(1 + \|y_2(t)\| + \|w\|) \\ &\leq d_2(1 + \|w\|),\end{aligned}$$

where $d_2 = d_1(1 + \|y_0^2\| + \int_0^T \|\dot{y}_2(s)\|ds)$.
Fix $(z_1, z_2) \in \mathcal{Z}_1 \times \mathcal{Z}_2$. Then, define the maps f_{1,z_1},f_{2,z_2} by

$$f_{1,z_1}(t,s,u,v) = f_1(t,s,u,v,z_1(s)), \ \ f_{2,z_2}(t,u,v) = f_2(t,u,v,z_2(t)),$$

for any $(t,s,u,v) \in I \times I \times H \times H$.
Clearly, by assumptions $(h^1_{f_1})$ and $(h^1_{f_2})$, the required measurability and continuity of f_{1,z_1} and f_{2,z_2} in Corollary 2.2 are fulfilled.

Since $\mathcal{Z}_1, \mathcal{Z}_2$ are compact sets in $\mathcal{C}(I, H)$, there exist nonnegative real constants τ_1, τ_2 such that $\mathcal{Z}_1 \subset \tau_1 \overline{B}_H$ and $\mathcal{Z}_2 \subset \tau_2 \overline{B}_H$. Then by (3.1), (3.3), for all $(t, s, u, v) \in I \times I \times H \times H$ one has

$$\begin{aligned}\|f_{1,z_1}(t, s, u, v)\| = \|f_1(t, s, u, v, z_1(s))\| &\leq m_1(1 + \|u\| + \|v\| + \|z_1(s)\|)\\ &\leq m_2(1 + \|u\| + \|v\|),\end{aligned}$$

and

$$\begin{aligned}\|f_{2,z_2}(t, u, v)\| = \|f_2(t, u, v, z_2(t))\| &\leq l_1(1 + \|u\| + \|v\| + \|z_2(t)\|)\\ &\leq l_2(1 + \|u\| + \|v\|),\end{aligned}$$

for nonnegative real constants m_2 and l_2.
Moreover, by (3.2) and (3.4), for some $\eta > 0$, there exist two nonnegative functions $\sigma_\eta(\cdot), \psi_\eta(\cdot) \in L^2(I, \mathbb{R})$ such that for all $(t, s) \in I \times I$ and any $u_1, v_1, u_2, v_2 \in \overline{B}_H[0, \eta]$, one has

$$\begin{aligned}\|f_{1,z_1}(t, s, u_1, v_1) - f_{1,z_1}(t, s, u_2, v_2)\| &= \|f_1(t, s, u_1, v_1, z_1(s)) - f_1(t, s, u_2, v_2, z_1(s))\|\\ &\leq \psi_\eta(t)(\|u_1 - u_2\| + \|v_1 - v_2\|),\end{aligned}$$

and

$$\begin{aligned}\|f_{2,z_2}(t, u_1, v_1) - f_{2,z_2}(t, u_2, v_2)\| &= \|f_2(t, u_1, v_1, z_2(t)) - f_2(t, u_2, v_2, z_2(t))\|\\ &\leq \sigma_\eta(t)(\|u_1 - u_2\| + \|v_1 - v_2\|).\end{aligned}$$

Then, all conditions of Corollary 2.2 are verified. The solution of problem (1.2) exists and is unique. ☐

Now, we prove the existence of optimal solutions to our minimization problem.

Theorem 3.2. *Assume that for any $(t, x) \in I \times H$, $B_1(t, x) : \mathrm{D}\,(B_1(t, x)) \subset H \to 2^H$ is a maximal monotone operator verifying $(H^1_{B_1})$-$(H^2_{B_1})$-$(H^3_{B_1})$. Assume that for any $(t, x) \in I \times H$, $B_2(t, x) : \mathrm{D}\,(B_2(t, x)) \subset H \to 2^H$ is a maximal monotone operator verifying $(H^1_{B_2})$-$(H^2_{B_2})$-$(H^3_{B_2})$.*
Let $f_1 : I \times I \times H \times H \times H \to H$ be a map such that assumptions $(h^1_{f_1})$-$(h^2_{f_1})$-$(h^3_{f_1})$ hold true. Let $f_2 : I \times H \times H \times H \to H$ be a map such that assumptions $(h^1_{f_2})$-$(h^2_{f_2})$-$(h^3_{f_2})$ hold true.
Let $\varphi : H \times H \to \mathbb{R}$ be lower semi continuous.
Then, the minimization problem

$$\min_{(y,z)\in\mathcal{Y}\times\mathcal{Z}} \varphi(u_{y,z}(T), x_{y,z}(T)), \tag{3.6}$$

where $(u_{y,z}, x_{y,z})$ is the unique solution associated to the controls y, z to problem (1.2), has an optimal solution.

Proof. First, note that the solution of problem (1.2) exists and is unique by Theorem 3.1.

Let $y_n = (y_{1,n}, y_{2,n})$ and $z_n = (z_{1,n}, z_{2,n})$ be minimizing sequences of problem (3.6), that is,

$$\lim_{n\to\infty} \varphi(u_n(T), x_n(T)) = \min_{(v,w)\in\mathcal{Y}\times\mathcal{Z}} \varphi(u_{v,w}(T), x_{v,w}(T))$$

where for each n, (u_n, x_n) is the unique solution to the following problem:

$$\begin{cases} -\dot{u}_n(t) \in B_1(t, y_{1,n}(t))u_n(t) + \displaystyle\int_0^t f_1(t, s, x_n(s), u_n(s), z_{1,n}(s))ds \quad \text{a.e. } t \in I, \\ -\dot{x}_n(t) \in B_2(t, y_{2,n}(t))x_n(t) + f_2(t, x_n(t), u_n(t), z_{2,n}(t)) \quad \text{a.e. } t \in I, \\ (y_0^1, y_0^2) = (y_{1,n}(0), y_{2,n}(0)), \\ z_n = (z_{1,n}, z_{2,n}) \in \mathcal{Z} = \mathcal{Z}_1 \times \mathcal{Z}_2, \\ y_n = (y_{1,n}, y_{2,n}) \in \mathcal{Y} = \mathcal{Y}_1 \times \mathcal{Y}_2, \\ (u_n(0), x_n(0)) = (u_0, x_0) \in \mathrm{D}\,(B_1(0, y_0^1)) \times \mathrm{D}\,(B_2(0, y_0^2)). \end{cases} \tag{3.7}$$

In view of (3.5), there exist $\overline{u}$, $\overline{x} \in W^{1,2}(I, H)$ such that

$$(u_n) \text{ pointwise converges to } \overline{u}, \tag{3.8}$$

$$(\dot{u}_n) \text{ weakly converges in } L^2(I, H) \text{ to } \dot{\overline{u}}, \tag{3.9}$$

and

$$(x_n) \text{ pointwise converges to } \overline{x}, \tag{3.10}$$

$$(\dot{x}_n) \text{ weakly converges in } L^2(I, H) \text{ to } \dot{\overline{x}}. \tag{3.11}$$

Remember that $\mathcal{Y}_1$ and $\mathcal{Y}_2$ are compact in $\mathcal{C}(I, H)$, extracting a subsequence (keeping the same notation of each sequence), one assumes that

$$(y_{1,n}) \text{ uniformly converges to } \overline{y}_1 \in \mathcal{Y}_1, \tag{3.12}$$

$$(y_{2,n}) \text{ uniformly converges to } \overline{y}_2 \in \mathcal{Y}_2. \tag{3.13}$$

Since $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are compact in $\mathcal{C}(I, H)$, extracting a subsequence (keeping the same notation of each sequence), one assumes that

$$(z_{1,n}) \text{ uniformly converges to } \overline{z}_1 \in \mathcal{Z}_1, \tag{3.14}$$

$$(z_{2,n}) \text{ uniformly converges to } \overline{z}_2 \in \mathcal{Z}_2. \tag{3.15}$$

Let for any n and any $t \in I$, $h_{1,n}(t) = \int_0^t f_1(t, s, x_n(s), u_n(s), z_{1,n}(s))ds$. Since $f_1(t, s, \cdot, \cdot, \cdot)$ is continuous, then by (3.8), (3.10), and (3.14), one gets

$$f_1(t, s, x_n(s), u_n(s), z_{1,n}(s)) \to f_1(t, s, \overline{x}(s), \overline{u}(s), \overline{z}_1(s)) \text{ as } n \to \infty.$$

Moreover, from $(h_{f_1}^2)$ and the fact that the sequences (u_n), (x_n), and $(z_{1,n})$ are bounded in $\mathcal{C}(I, H)$, then the Lebesgue dominated convergence theorem gives

$$h_{1,n}(t) \to h(t), \quad \text{as} \quad n \to \infty,$$

with $h(t) = \int_0^t f_1(t, s, \overline{x}(s), \overline{u}(s), \overline{z}_1(s))ds$, $t \in I$. Once more by $(h_{f_1}^2)$ and the boundedness of the sequences (u_n), (x_n) and $(z_{1,n})$, then

$$(h_{1,n}(\cdot)) \text{ converges in } L^2(I, H) \text{ to } h(\cdot), \tag{3.16}$$

using the Lebesgue dominated convergence theorem.

Thanks to (3.8), (3.10) and (3.15) and the continuity of $f_2(t, \cdot, \cdot, \cdot)$ for a.e. $t \in I$

$$f_2(t, x_n(t), u_n(t), z_{2,n}(t)) \to f_2(t, \overline{x}(t), \overline{u}(t), \overline{z}_2(t)) \text{ as } n \to \infty,$$

along with $(h_{f_2}^2)$ and the fact that (x_n), (u_n) and $(z_{2,n})$ are bounded, then the Lebesgue dominated convergence theorem gives

$$(f_2(\cdot, x_n(\cdot), u_n(\cdot), z_{2,n}(\cdot))) \text{ converges in } L^2(I, H) \text{ to } f_2(\cdot, \overline{x}(\cdot), \overline{u}(\cdot), \overline{z}_2(\cdot)). \tag{3.17}$$

Now, since φ is lower semi-continuous, it results

$$\liminf_{n\to\infty} \varphi(u_n(T), x_n(T)) \geq \varphi(\overline{u}(T), \overline{x}(T)).$$

Thus, one gets

$$\inf_{(v,w)\in\mathcal{Y}\times\mathcal{Z}} \varphi(u_{v,w}(T), x_{v,w}(T)) = \varphi(\overline{u}(T), \overline{x}(T)).$$

Finally, let us verify that

$$\begin{cases} -\dot{\overline{u}}(t) \in B_1(t, \overline{y}_1(t))\overline{u}(t) + \displaystyle\int_0^t f_1(t, s, \overline{x}(s), \overline{u}(s), \overline{z}_1(s))ds \quad \text{a.e. } t \in I, \\ -\dot{\overline{x}}(t) \in B_2(t, \overline{y}_2(t))\overline{x}(t) + f_2(t, \overline{x}(t), \overline{u}(t), \overline{z}_2(t)) \quad \text{a.e. } t \in I, \\ (y_0^1, y_0^2) = (\overline{y}_1(0), \overline{y}_2(0)), \\ \overline{z} = (\overline{z}_1, \overline{z}_2) \in \mathcal{Z}_1 \times \mathcal{Z}_2, \\ \overline{y} = (\overline{y}_1, \overline{y}_2) \in \mathcal{Y}_1 \times \mathcal{Y}_2, \\ (\overline{u}(0), \overline{x}(0)) = (u_0, x_0) \in \mathrm{D}\left(B_1(0, y_0^1)\right) \times \mathrm{D}\left(B_2(0, y_0^2)\right). \end{cases} \tag{3.18}$$

From (3.7) and the preceding convergence modes (see (3.8), (3.9), (3.10), (3.11), (3.12), (3.13), (3.16) and (3.17)), we argue as in **Part 3** of the proof of Theorem 2.1 to show the inclusions in (3.18).

As the solution of (3.18) is unique (see Theorem 3.1), one concludes that $(\overline{u}, \overline{x}) = (u_{\overline{y},\overline{z}}, x_{\overline{y},\overline{z}})$ is the unique solution to problem (1.2) where, $\overline{y} = (\overline{y}_1, \overline{y}_2)$ and $\overline{z} = (\overline{z}_1, \overline{z}_2)$. The proof of the theorem is therefore finished. □

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[1] LAOTI Laboratory, Faculty of Exact Sciences and Informatics, University of Jijel, 18000 Jijel, Algeria
Email address: nadjibaabdi@gmail.com;

[2] LMPA Laboratory, Faculty of Exact Sciences and Informatics, University of Jijel, 18000 Jijel, Algeria
Email address: soumiasaidi44@gmail.com