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POSNER'S FIRST THEOREM FOR PRIME MODULES

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Abstract. Let R be a ring, let M be a left R-module, and let $\tau : M \to M$ and $\delta : R \to R$ be additive maps. We say that τ is a generalized derivation relative to δ if $\tau(am) = a\tau(m) + \delta(a)m$ for all $a \in R$ and $m \in M$. In this paper, we provide a generalization of Posner's first theorem to generalized derivations on 2-torsion free prime modules. We also obtain a result of this generalization in connection with derivations acting on left ideals of prime rings. Moreover, we extend some previous results related to Posner's first theorem. Furthermore, as an application of our main result, we examine Posner's first theorem for a certain class of derivations on trivial extension rings under some suitable conditions.

1. Introduction

Throughout this paper, all rings are associative with unity and all modules are unital. Let R be a ring. An additive mapping $\delta : R \to R$ is said to be a *derivation* if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$. Derivation is an important class of maps by which one can study the structure of the rings. The product of two derivations is not necessarily a derivation. Thus, the following question is worth studying: If the multiplication of two derivations is again a derivation, what is the structure of their multiplication derivations? Firstly, Posner [14] showed that if the product of two derivations on a prime ring with characteristic other than two is a derivation, then one of them should be zero. This result is known as the Posner's first theorem, which has already been generalized in several directions, among which one can point to acting the derivations on one sided ideals. We refer the reader to [1, 3, 8, 9] and the references therein for more details.

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Brešar [2] has defined the concept of a generalized derivation as follows: An additive mapping $\tau : R \to R$ is called a *generalized derivation* if there exists a derivation $\delta : R \to R$ such that $\tau(ab) = \tau(a)b + a\delta(b)$ for all $a, b \in R$. Obviously, every derivation is a generalized derivation. Moreover, the generalized derivation τ with $\delta = 0$ covers the concept of *left multiplier*, that is, an additive map $\tau : R \to R$ satisfying $\tau(ab) = \tau(a)b$ for all $a, b \in R$. Also, we can define *left generalized derivation* by $\tau(ab) = a\tau(b) + \delta(a)b$ for all $a, b \in R$, which covers the concepts of derivations and right multipliers both. Hvala [7] studied the Posner's first theorem for generalized derivations on 2-torsion free prime rings. Motivated by the definition of generalized derivation on rings, in [5], generalized derivation on left modules over a complex algebra has been defined as follows: Suppose that A is a complex algebra and that M is a left A-module. A linear map $\tau : M \to M$ is called a *generalized derivation* relative to the linear derivation $\delta : A \to A$ if

$$\tau(am) = a\tau(m) + \delta(a)m \quad (a \in A, m \in M). \tag{1.1}$$

Some results on generalized derivations and special versions of Posner's first theorem on prime modules can be found in [5].

In this paper, we generalize the definition of generalized derivation on modules as follows: Let R be a ring, let M be a left R-module, and let $\delta : R \to R$ be an additive map. An additive map $\tau : M \to M$ is called a *generalized derivation* relative to δ if

$$\tau(am) = a\tau(m) + \delta(a)m \quad (a \in R, m \in M).$$

It is clear that if δ is a derivation, then the concepts of the generalized derivations given above and the one given in (1.1) coincide. Moreover, if we consider R as a left R-module and δ is a derivation on R, then τ is a generalized derivation on R in the sense defined in [2]. So this definition actually generalizes the previous definitions of generalized derivation. We state all of our results for left modules. Right module analogues can be obtained with the same argument. The main result of this paper is a generalization of Posner's first theorem for generalized derivations on 2-torsion free prime left modules. In fact, considering every left ideal of R as a left R-module, by the main result of this paper, we will obtain some results about Posner's first theorem on left ideals of R. Also, we generalize some previous results on the product of derivations on prime rings to the product of generalized derivations on prime modules.

Noting that Posner's first theorem, its generalizations, and the related results are on prime rings, the question that naturally arises is how does the Posner's first theorem work for nonprime rings? Trivial extensions are examples of nonprime rings. We investigate Posner's first theorem for the derivations on trivial extension rings under mild conditions, as an application of the main result of this paper.

This paper is organized as follows. In Sec 2, some necessary definitions and preliminaries are provided. In Sec 3, we present the main result and some other results related to the Posner's first theorem on prime modules. The last section is devoted to investigating the Posner's first theorem on trivial extension rings.

2. Preliminaries

From this point up to the last section, R is a unitary ring and M is a left R-module. Let us first recall the definition of a generalized derivation.

Definition 2.1. Let $\delta : R \to R$ be an additive map. An additive map $\tau : M \to M$ is called a *generalized derivation* relative to δ if the following identity holds:

$$\tau(am) = a\tau(m) + \delta(a)m \quad (a \in R, m \in M).$$

In the following, examples (i)-(iii) show that this definition extends some important classes of mappings, and the last one is a counterexample.

Example 2.2.

- (i) Let $\tau : M \to M$ be an R-module homomorphism. Then τ is a generalized derivation relative to the zero map.
- (ii) Let $\delta : R \to R$ be a derivation. Then δ is a generalized derivation relative to itself.
- (iii) Let $a \in R$. The additive map $\tau_a : M \to M$ defined by $\tau_a(m) = am$ is a generalized derivation relative to the inner derivation $I_a : R \to R$, where $I_a(b) = ab - ba$.
- (iv) The following operation makes M into a left $R \times R$-module:

$$(a, b)m = am \quad (a, b \in R, m \in M).$$

Define $\tau : M \to M$ by $\tau(m) = m$ and $\delta : R\times R \to R\times R$ by $\delta(a, b) = (0, a)$. The additive mapping τ is a generalized derivation relative to δ, but δ is not a derivation on $R \times R$.

Example 2.2(iv) shows that if τ is a generalized derivation relative to δ, then δ is not necessarily a derivation. The following remark provides a sufficient condition on M under which δ is a derivation.

Remark 2.3. Let $\tau : M \to M$ be a generalized derivation relative to $\delta : R \to R$. If M is *faithful*, that is,

$$lann_R M = \{r \in R \,:\, rM = (0)\} = (0),$$

then δ is a derivation: Assume that $r_1, r_2 \in R$ and $m \in M$. Then

$$\tau(r_1r_2m) = r_1r_2\tau(m) + \delta(r_1r_2)m.$$

On the other hand,

$$\begin{aligned}\tau(r_1r_2m) &= r_1\tau(r_2m) + \delta(r_1)r_2m \\ &= r_1r_2\tau(m) + r_1\delta(r_2)m + \delta(r_1)r_2m.\end{aligned}$$

Comparing these two identities, we have

$$(\delta(r_1r_2) - r_1\delta(r_2) - \delta(r_1)r_2)m = 0,$$

for all $m \in M$, concluding that δ is a derivation.

Recall that the ring of all R-module endomorphisms on M is denoted by $End_R M$, and M is called 2-*torsion free* if $2m = 0$ ($m \in M$) implies $m = 0$.

Remark 2.4. The subset $I = lann_R M$ is an ideal of R, and if M is 2-torsion free, then the quotient ring R/I is 2-torsion free: let $a \in R$ be such that $2(a+I) = I$. Hence $2a \in I$ and, for all $m \in M$, $2am = 0$. Since M is 2-torsion free, we have $am = 0$ for all $m \in M$, and hence $a \in I$, showing that R/I is 2-torsion free.

Let P be a submodule of M. Then the quotient

$$(P : M) = \{a \in R \,:\, aM \subseteq P\}$$

is an ideal of R. If $P = (0)$, then obviously $((0) : M) = lann_R M$. More precisely, we have

$$(P : M) = lann_R(M/P).$$

A proper submodule P of M is called a *prime submodule* if for every $a \in R$ and $m \in M$, the inclusion $aRm \subseteq P$ implies that $m \in P$ or $a \in (P : M)$. We say that M is a *prime module* if (0) is a prime submodule of M. If P is a prime submodule of M, then $(P : M)$ is a prime ideal of R. Furthermore, P is a prime submodule of M if and only if M/P is prime as a left R-module. We refer the reader to [11, 10, 12, 13, 15] for more information.

Let M be an R-bimodule. Then $R \times M$ is an abelian group, and together with the product defined by

$$(a, m)(b, n) = (ab, an + mb) \quad (a, b \in R,\ m, n \in M),$$

it is a ring with unity $(1, 0)$, which is called the *trivial extension* of R by M and denoted by $T(R, M)$. Let $I = (0) \times M$. Then I is an ideal of $T(R, M)$ such that $I^2 = (0)$. So $T(R, M)$ is not a prime ring.

3. Generalized derivations on prime modules $\mathcal{T}, \mathfrak{A}$

In this section, we present the main results of this paper. Throughout this section, R is a unitary ring and M is a left R-module.

The next theorem is a generalization of Posner's first theorem for generalized derivations on prime modules.

Theorem 3.1. *Suppose that M is a 2-torsion free prime module. Let δ_1, δ_2 be additive maps on R and let $\tau_1, \tau_2 : M \to M$ be generalized derivations relative to δ_1, δ_2, respectively. Then $\tau_1\tau_2$ is a generalized derivation relative to the additive map $\delta_1\delta_2$ if and only if one of the following conditions holds:*

(i) $\tau_1 = 0$;
(ii) $\tau_2 = 0$;
(iii) $\tau_1 \neq 0, \tau_2 \neq 0$, *and* $\tau_1, \tau_2 \in End_R M$.

Proof. Let $\tau_1\tau_2$ be a generalized derivation relative to $\delta_1\delta_2$ and put $I = lann_R M$. Since M is a prime module, it follows that I is a prime ideal of R, and R/I is a prime ring. Moreover, M is a left R/I-module by the following module action:

$$(a + I)m = am \quad (a \in R, m \in M).$$

Let $a \in I$. Then for all $m \in M$ and $i = 1, 2$, we have

$$\tau_i(am) = a\tau_i(m) + \delta_i(a)m.$$

So $\delta_i(a)m = 0$, and hence $\delta_i(I) \subseteq I$. Define $\tilde{\delta_i}: R/I \to R/I$, $(i = 1, 2)$ by

$$\tilde{\delta_i}(a + I) = \delta_i(a) + I.$$

Since $\delta_i(I) \subseteq I$, it follows that each $\tilde{\delta_i}$ is well defined. Also, each $\tilde{\delta_i}$ is an additive map. Now, for $i = 1, 2$, we see that

$$\begin{aligned} \tau_i((a + I)m) &= \tau_i(am) \\ &= a\tau_i(m) + \delta_i(a)m \\ &= (a + I)\tau_i(m) + (\delta_i(a) + I)m \\ &= (a + I)\tau_i(m) + \tilde{\delta_i}(a + I)m \end{aligned}$$

for all $a \in R$ and $m \in M$. So each τ_i is a generalized derivation relative to $\tilde{\delta_i}$, and since $lann_{R/I}M = (0)$, from Remark 2.3 it follows that each $\tilde{\delta_i}$ is derivation on R/I. On the other hand, it can be easily checked that $\widetilde{\delta_1\delta_2} = \tilde{\delta_1}\tilde{\delta_2}$. From the fact that $\tau_1\tau_2$ is a generalized derivation relative to $\delta_1\delta_2$, using a similar argument as above, it follows that $\tau_1\tau_2$ is a generalized derivation relative to $\widetilde{\delta_1\delta_2} = \tilde{\delta_1}\tilde{\delta_2}$ (in this case, M is a left R/I-module). So $\tilde{\delta_1}\tilde{\delta_2}$ is a derivation on R/I, because M is a faithful left R/I-module.

The ring R/I is prime and by Remark 2.4, it is also 2-torsion free. In addition, $\tilde{\delta_1}\tilde{\delta_2}$ is a derivation on R/I. Thus, by Posner's first theorem, $\tilde{\delta_1} = 0$ or $\tilde{\delta_2} = 0$.

Let $\tilde{\delta_1} = 0$. In this case, according to the definition of $\tilde{\delta_1}$, we have $\delta_1(R) \subseteq I$. So

$$\tau_1(am) = a\tau_1(m) \quad (a \in R, m \in M).$$

This means that $\tau_1 \in End_R M$. Hence

$$\begin{aligned} \tau_1\tau_2(am) &= \tau_1(a\tau_2(m) + \delta_2(a)m) \\ &= a\tau_1\tau_2(m) + \delta_2(a)\tau_1(m). \end{aligned}$$

On the other hand, our assumptions and the fact that $\delta_1(R) \subseteq I$ imply that

$$\begin{aligned} \tau_1\tau_2(am) &= a\tau_1\tau_2(m) + \delta_1\delta_2(a)m \\ &= a\tau_1\tau_2(m). \end{aligned}$$

Comparing these two identities, we arrive at

$$\delta_2(a)\tau_1(m) = 0 \quad (a \in R, m \in M).$$

Thus for all $a, b \in R$ and $m \in M$, we have

$$\delta_2(a)b\tau_1(m) = \delta_2(a)\tau_1(bm) = 0.$$

Therefore $\delta_2(a)R\tau_1(m) = 0$ for all $a \in R$ and $m \in M$. Primeness of M implies that either $\delta_2(a)M = (0)$ or $\tau_1(m) = 0$ for all $a \in R, m \in M$. So either $\tau_1 = 0$ or $\tau_1 \neq 0$ and $\delta_2(a)M = (0)$ for all $a \in R$, that is, $\tau_1 = 0$ or $\tau_1 \neq 0$ and $\tau_2 \in End_R M$.

By a similar argument as above, we can show that if $\tilde{\delta_2} = 0$, then either $\tau_2 = 0$ or $\tau_2 \neq 0$ and $\tau_1 \in End_R M$.

Consequently, one of the possibilities (i), (ii), or (iii) holds true.

Conversely, assume that the case (ii) happens. Then $\delta_2(R) \subseteq lann_R M$. By a similar argument as given above, we can prove that $\delta_1(lann_R M) \subseteq lann_R M$. Consequently, $\tau_1\tau_2 = 0$, and we have

$$0 = \tau_1\tau_2(am) = a\tau_1\tau_2(m) + \delta_1\delta_2(a)m$$

for all $a \in R$ and $m \in M$. This means that $\tau_1\tau_2$ is a generalized derivation relative to $\delta_1\delta_2$. If we have any of the cases (i) or (iii), a similar argument shows that $\tau_1\tau_2$ is a generalized derivation relative to $\delta_1\delta_2$. □

Suppose that R is a 2-torsion free prime ring and that δ_1, δ_2 are derivations on R. If we consider R as a left R -module, then R is a prime module and δ_1, δ_2 are generalized derivations relative to δ_1, δ_2, respectively. Now, if $\delta_1\delta_2$ is a generalized derivation relative to the additive mapping $\delta_1\delta_2$, then by the previous theorem, we should have either $\delta_1 = 0$, $\delta_2 = 0$, or $\delta_1, \delta_2 \in End_R(R)$. Since R is a prime ring and δ_1, δ_2 are derivations, we should have $\delta_1 = 0$ or $\delta_2 = 0$, which is the Posner's first theorem. Hence Theorem 3.1 is a generalization of Posner's first theorem on prime modules.

In the next corollary, we consider the iterate of generalized derivations and module endomorphisms.

Corollary 3.2. *Suppose that M is a 2-torsion free prime module, that $\tau : M \to M$ is a generalized derivation relative to additive map δ, and that $\phi \in End_R M$. Then $\tau\phi \in End_R M$ if and only if one of the following conditions holds:*

(i) $\tau = 0$;
(ii) $\phi = 0$;
(iii) $\tau \neq 0, \phi \neq 0$ *and* $\tau \in End_R M$.

Proof. Since $\phi \in End_R M$, it follows that ϕ is a generalized derivation relative to $\gamma = 0$. Then $\tau\phi \in End_R M$ is a generalized derivation relative to $\delta\gamma = 0$. Now, the result follows from Theorem 3.1. □

One notes that [5, Theorem 3.7] is a special case of the corollary above. So Theorem 3.1 is also a generalization of [5, Theorem 3.7].

In the next corollary, we present a generalization of Posner's first theorem to derivations acting on left ideals.

Corollary 3.3. *Suppose that R is a 2-torsion free prime ring and that L is a nonzero left ideal of R. Let $\delta_1, \delta_2 : R \to R$ be derivations such that $\delta_i(L) \subseteq L$ $(i = 1, 2)$ and for all $a \in R$, $x \in L$*

$$\delta_1\delta_2(ax) = a\delta_1\delta_2(x) + \delta_1\delta_2(a)x.$$

Then either $\delta_1 = 0$ or $\delta_2 = 0$.

Proof. Let $a \in R$ and $x \in L$ be such that $aRx = (0)$. Since R is prime, we have $a = 0$ or $x = 0$. So either $aL = (0)$ or $x = 0$, concluding that L is prime as a left R-module. Define the additive mappings $\tau_i : L \to L$ $(i = 1, 2)$ by $\tau_i = \delta_i|_L$. Then

$$\begin{aligned}\tau_i(ax) &= \delta_i(ax)\\ &= a\delta_i(x) + \delta_i(a)x\\ &= a\tau_i(x) + \delta_i(a)x\end{aligned}$$

for all $a \in R$ and $x \in L$, $i = 1, 2$. So each τ_i is a generalized derivation relative to δ_i. According to the assumption,

$$\begin{aligned}\tau_1\tau_2(ax) &= \delta_1\delta_2(ax)\\ &= a\delta_1\delta_2(x) + \delta_1\delta_2(a)x\\ &= a\tau_1\tau_2(x) + \delta_1\delta_2(a)x\end{aligned}$$

for all $a \in R$, $x \in L$. Hence, $\tau_1\tau_2$ is a generalized derivation relative to (the additive map) $\delta_1\delta_2$. Now all conditions of Theorem 3.1 hold for τ_1 and τ_2 on the 2-torsion free left R-module L. So either $\tau_1 = 0$, $\tau_2 = 0$, or $\tau_1, \tau_2 \neq 0$ and $\tau_1, \tau_2 \in End_R L$. If $\tau_1, \tau_2 \neq 0$ and $\tau_1, \tau_2 \in End_R L$, then for $i = 1, 2$ we have

$$\tau_i(ax) = a\tau_i(x) = a\delta_i(x).$$

On the other hand,

$$\begin{aligned}\tau_i(ax) &= \delta_i(ax)\\ &= a\delta_i(x) + \delta_i(a)x,\end{aligned}$$

and hence $\delta_i(a)x = 0$ $(a \in R, x \in L)$. Thus for all $a \in R$ and $0 \neq x \in L$, we have

$$\delta_i(a)Rx = 0.$$

The primeness of R shows that $\delta_i = 0$. So $\tau_i = 0$ for $i = 1, 2$, a contradiction. Therefore, the third case is impossible. If $\tau_1 = 0$ or $\tau_2 = 0$, then a similar argument shows that either $\delta_1 = 0$ or $\delta_2 = 0$. □

Creedon [4, Theorem 2] proved that if δ_1,δ_2 are derivations on a ring R such that $\delta_1\delta_2(R) \subseteq P$, where P is a prime ideal for which the characteristic of R/P is not 2, then $\delta_1(R) \subseteq P$ or $\delta_2(R) \subseteq P$. If $P = (0)$, then obviously, the Creedon's result is just the Posner's first theorem. In the next theorem, we generalize the Creedon's result to left modules.

Theorem 3.4. *Suppose that M is a left R-module and that P is a prime submodule of M such that M/P is 2-torsion free. Let δ_1, δ_2 be derivations on R and let τ_1, τ_2 be generalized derivations relative to δ_1, δ_2, respectively, such that $\tau_1\tau_2(M) \subseteq P$ and $\delta_1\delta_2(R) \subseteq (P : M)$. Then one of the following conditions holds:*

(i) $\tau_1(M) \subseteq P$;
(ii) $\tau_2(M) \subseteq P$;
(iii) $\tau_1(M) \nsubseteq P, \tau_2(M) \nsubseteq P$ *and* $\delta_1(R) \subseteq (P : M), \delta_2(R) \subseteq (P : M)$.

Proof. From the fact that $lann_R(M/P) = (P : M)$ and Remark 2.4, it follows that $R/(P : M)$ is 2-torsion free. Also, $(P : M)$ is a prime ideal, since P is a prime submodule. Now, according to [4, Theorem 2], we have

$$\delta_1(R) \subseteq (P : M) \quad \text{or} \quad \delta_2(R) \subseteq (P : M).$$

Let $\delta_1(R) \subseteq (P : M)$. Then

$$\begin{aligned}\tau_1\tau_2(am) =& a\tau_1\tau_2(m) + \delta_1(a)\tau_2(m)\\ &+ \delta_2(a)\tau_1(m) + \delta_1\delta_2(a)m\end{aligned}$$

for all $a \in R$, $m \in M$. By hypothesis, we conclude that $\delta_2(a)\tau_1(m) \in P$ $(a \in R, m \in M)$. Now,

$$\delta_2(a)\tau_1(bm) = \delta_2(a)b\tau_1(m) + \delta_2(a)\delta_1(b)m$$

for all $a, b \in R$, $m \in M$. Since $\delta_1(b)m \in P$ and $\delta_2(a)\tau_1(bm) \in P$, it follows that $\delta_2(a)b\tau_1(m) \in P$ for all $a, b \in R$, $m \in M$. So

$$\delta_2(a)R\tau_1(m) \subseteq P$$

for all $a \in R$, $m \in M$. Since P is prime, we have two possibilities: $\tau_1(M) \subseteq P$ or $\tau_1(M) \nsubseteq P$, $\delta_2(a)M \subseteq P$ $(a \in R)$, that is, either $\tau_1(M) \subseteq P$ or $\tau_1(M) \nsubseteq P$ and $\delta_2(R) \subseteq (P : M)$.

If $\delta_2(R) \subseteq (P : M)$, then a similar argument shows that either $\tau_2(M) \subseteq P$ or $\tau_2(M) \nsubseteq P$ and $\delta_1(R) \subseteq (P : M)$.

□

Consider the ring R as a left R-module, and let P be a prime ideal not of characteristic 2. Clearly P is a prime submodule of R. If δ_1, δ_2 are derivations on R such that $\delta_1\delta_2(R) \subseteq P$, then δ_1, δ_2 are generalized derivations relative to δ_1, δ_2, respectively, and $\delta_1\delta_2(R) \subseteq (P : R) = P$. Thus the conditions of the previous theorem are fulfilled on R as a left R-module and derivations δ_1, δ_2. So either $\delta_1(R) \subseteq P$ or $\delta_2(R) \subseteq P$ (third possibility can not happen). This shows that Theorem 3.4 is a generalization of [4, Theorem 2]. Also, [5, Proposition 3.6(ii)] is a special case of Theorem 3.4.

As a consequence of Theorem 3.4, we have the following corollary.

Corollary 3.5. *Suppose that M is a 2-torsion free prime left R-module, that $\tau : M \to M$ is a generalized derivation relative to the derivation $\delta : R \to R$, and that $\phi \in End_R M$. Let $\tau\phi = 0$. Then either $\tau = 0$, $\phi = 0$ or $\tau \neq 0, \phi \neq 0, \tau \in End_R M$.*

Proof. The endomorphism ϕ is a generalized derivation relative to the derivation $\gamma = 0$. Let $P = (0)$. In this case, $\delta\gamma(R) = (0) \subseteq lann_R M$ and $\tau\phi(M) = (0)$. Now, the conditions of Theorem 3.4 are satisfied, so the result will be obtained. □

A special case of Corollary 3.5 is [5, Proposition 3.6(iii)]. Thus Theorem 3.4 is a generalization of [5, Proposition 3.6(iii)].

4. Product of derivations on trivial extensions

In this section, we examine Posner's first theorem for a certain class of derivations on trivial extensions. Throughout this section, R is a unitary ring and M is an R-bimodule. Note that the concept of derivation can be generalized as follows: An additive mapping $\delta : R \to M$ is called a derivation if $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in R$.

In the following lemma, which is proved in [6], the structure of derivations on trivial extensions has been described.

Lemma 4.1. ([6, Theorem 2.1]) *Let Δ be an additive map on the trivial extension $T(R,M)$. Then Δ is a derivation if and only if there exist additive maps $\delta : R \to R$, $\gamma : R \to M$, $\lambda : M \to R$ and $\tau : M \to M$ such that*

$$\Delta(r,m) = (\delta(r) + \lambda(m), \gamma(r) + \tau(m))$$

and

(i) *δ and γ are derivations;*

(ii) *τ satisfies*

$$\tau(am) = a\tau(m) + \delta(a)m \quad \text{and} \quad \tau(ma) = \tau(m)a + m\delta(a)$$

for all $a \in R$ and $m \in M$;

(iii) *λ is an R-bimodule homomorphism satisfying*

$$\lambda(m)m' + m\lambda(m') = 0$$

for all $m, m' \in M$.

According to part (ii) of the above lemma, τ is a generalized derivation relative to δ on M, both as a left R-module and as a right R-module.

Remark 4.2. In view of the previous lemma, if $\delta : R \to R$ and $\tau : M \to M$ are additive maps, then the additive map $\Delta : T(R,M) \to T(R,M)$ defined by

$$\Delta((a,m)) = (\delta(a), \tau(m))$$

is a derivation on $T(R,M)$ if and only if δ is a derivation and τ is a generalized derivation relative to δ, both from the left and right. We investigate the Posner's first theorem for this class of derivations on $T(R,M)$.

First, in the following example, we show that the product of two nonzero derivations on $T(R,M)$ can be a derivation.

Example 4.3. Define $\Delta : T(R,M) \to T(R,M)$ by

$$\Delta((a,m)) = (0,m).$$

It is easily verified that Δ is a derivation on $T(R,M)$. In addition,

$$\Delta^2((a,m)) = (0,m) = \Delta(0,m).$$

So Δ^2 is also a derivation on $T(R,M)$. Indeed $\Delta \neq 0$.

In the next theorem, we consider derivations on $T(R,M)$ as mentioned in Remark 4.2, and we examine Posner's first theorem for this class of derivations.

Theorem 4.4. *Suppose that $\Delta_i((a,m)) : T(R,M) \to T(R,M)$ $(i = 1,2)$ are derivations given by*

$$\Delta_i((a,m)) = (\delta_i(a), \tau_i(m)),$$

where each δ_i is a derivation on R and τ_i is a generalized derivation relative to δ_i on M both from the left and right. Let M be a faithful 2-torsion free prime left R-module. Then $\Delta_1\Delta_2$ is a derivation on $T(R,M)$ if and only if one of the following conditions holds:

(i) *$\Delta_1 = 0$;*

(ii) *$\Delta_2 = 0$;*

(iii) $\Delta_i((a,m)) = (0, \tau_i(m))$ $(i = 1,2)$, *where each* τ_i *is a nonzero* R*-bimodule endomorphism on* M.

Proof. We have

$$\begin{aligned}\Delta_1\Delta_2((a,m)) &= \Delta_1((\delta_2(a), \tau_2(m)) \\ &= (\delta_1\delta_2(a), \tau_1\tau_2(m)) \quad (a \in R, m \in M).\end{aligned}$$

Suppose that $\Delta_1\Delta_2$ is a derivation. By Remark 4.2, $\delta_1\delta_2$ is a derivation and $\tau_1\tau_2$ is a generalized derivation relative to $\delta_1\delta_2$ on M both as a left and as a right R-module. On the other hand, M is a 2-torsion free prime left R-module. Since conditions of Theorem 3.1 are satisfied by τ_1, τ_2, we have one of the following conditions:

(i) $\tau_1 = 0$;
(ii) $\tau_2 = 0$;
(iii) $\tau_1 \neq 0, \tau_2 \neq 0$ and $\tau_1, \tau_2 \in End_R M$.

(i) Let $\tau_1 = 0$. Then

$$0 = \tau_1(am) = a\tau_1(m) + \delta_1(a)m,$$

so that $\delta_1(a)m = 0$ for all $a \in R$ and $m \in M$. Since M is faithful as a left R-module, we conclude that $\delta_1 = 0$. Therefore, $\Delta_1 = 0$.

(ii) By a similar argument as in part (i), we can prove that $\Delta_2 = 0$.

(iii) Since $\tau_i \in End_R M$ $(i = 1,2)$, it follows that

$$\tau_i(am) = a\tau_i(m),$$

and on the other hand,

$$\tau_i(am) = a\tau_i(m) + \delta_i(a)m$$

for all $a \in R$ and $m \in M$. Comparing these two identities, we obtain $\delta_i(a)m = 0$ for all $a \in R$ and $m \in M$. The faithfulness of M as a left R-module implies that $\delta_i = 0$ $(i = 1,2)$. Now

$$\tau_i(ma) = \tau_i(m)a + m\delta_i(a) = \tau_i(m)a$$

for all $a \in R$ and $m \in M$. This means each τ_i is also a right R-module endomorphism on M. Therefore, each τ_i is an R-bimodule endomorphism on M. Since in this case $\tau_i \neq 0$ and $\delta_i = 0$ $(i = 1,2)$, we have

$$\Delta_i((a,m)) = (0, \tau_i(m)) \neq 0 \quad (a \in R, m \in M).$$

The converse is trivial. □

If R is a 2-torsion free prime ring, then R is a 2-torsion free prime left R-module, and obviously it is also faithful. So Theorem 4.4 holds true for $T(R,R)$, where R is a 2-torsion free prime ring.

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