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COMPLETE CLASSIFICATION OF NOETHER CONSERVATION LAWS FOR A PHYSICALLY SIGNIFICANT STATIONARY KALUZA–KLEIN PERFECT FLUID SOLUTION

FATEMEH AHANGARI

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Abstract. Steady rigid rotation of a fluid in general relativity has been remarkably tackled by many authors specifically after Godel proposed relativistic model of a rotating dust universe. Considering the fact that stationary Kaluza–Klein perfect fluid models in standard Einstein theory are not available in literature, the significance of obtaining and analyzing such solutions in order to investigate the effects of dimensionality on the different physical parameters is undoubtedly undeniable. In this paper, the problem of symmetries and conservation laws for some specific solutions of Kaluza–Klein field equations for stationary symmetric fluid models in standard Einstein theory is exhaustively analyzed. For this purpose, a physically viable stationary Kaluza–Klein perfect fluid solution is considered, and the corresponding point generators of one parameter Lie groups of transformations that leave invariant the action integral associated to the Lagrangian, namely, Noether symmetries, are computed. A brief discussion regarding the structure of the Lie algebra of Noether symmetries from the algebraic point of view is presented. Moreover, a complete classification of the resulted Noether symmetry subalgebras is proposed by constructing an optimal system of one-dimensional subalgebras via the adjoint representation. Besides, the Killing vector fields of our analyzed geodesic Lagrangian are totally determined. Significantly, all the corresponding conservation laws of the Euler–Lagrange (geodesic) equations concluded from the obtained Noether symmetries are comprehensively calculated.

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1. Introduction and preliminaries

The original Kaluza–Klein theory can be regarded as one of the first endeavors to unify two of the fundamental forces of nature, namely gravitation and electromagnetism. The connections among Minkowski's four-dimensional space-time and Maxwell's unification of electricity and magnetism, stimulated Nordestrom [24] in 1914 (English translation in [1]) and independently Kaluza [14] in 1921 (English translation in [1, 9, 22]) to demonstrate that five-dimensional general relativity comprises both Einstein's four-dimensional gravity and Maxwell's electromagnetic (EM) field. Nevertheless, they imposed an artificial constraint of no dependence on fifth coordinate known as the cylinder condition. In Kaluza's original paper, the obtained equations are then divided into three distinct sets. One of which is corresponding to the Einstein four-dimensional field equations for gravitation. Another is equivalent to Maxwell's equations for the EM field. A third one characterizes a scalar field. In 1926, Klein [16] (English translation in [1, 9, 22]) elaborated a five-dimensional extension of general relativity but with some important distinctions. Indeed, he proposed a physical basis to avoid Kaluza's cylinder condition in the compactification of the fifth dimension. Nowadays, this approach is extensively applied in higher-dimensional generalizations to contain weak and strong interactions. In addition, Klein illustrated that Kaluza's cylinder condition would appear inherently whenever the fifth coordinate had **(a):** a circular topology in which case physical field would periodically depend on it and hence could be Fourier-expanded; and **(b):** a small enough, that is, compactified scale in which case the energies of all Fourier manners above the ground state could be made extremely high as to be not observable. Consequently, physics would be practically independent of Kaluza's fifth dimension, as desired (refer to [28] for more details). The three main features of models discussed above are as follows: **(1):** Matter in dimension four can be regarded as a manifestation of pure geometry in dimension five, that is, no explicit energy momentum tensor ${}^{(4+d)}T_{AB}$ is required. Besides, the higher-dimensional Einstein tensor ${}^{(4+d)}G_{AB}$, that is, the metric and its derivatives, totally comprises the gravitational field as well as the electromagnetic and Yang-Mills fields. **(2):** The higher-dimensional theory is a minimal extension of general relativity in the sense that there is not any modification to the mathematical structure of Einstein's theory. Nevertheless, the only difference is that the indices instead of 0 to 3 run over 0 to $(3+d)$. **(3):** Physics depends only on the first four coordinates; that is, they are a priori cylindrical.

Taking into account the feature **(2)** of Kaluza's approach, the Christoffel symbols $\hat{\Gamma}^C_{AB}$, the Ricci tensor $\hat{R}_{AB}$, and the Ricci scalar $\hat{R}$ are defined as follows [28]:

$$\begin{cases} \hat{\Gamma}^C_{AB} = \dfrac{1}{2}\hat{g}^{CD}\Big(\partial_A \hat{g}_{DB} + \partial_B \hat{g}_{DA} - \partial_D \hat{g}_{AB}\Big), \\ \hat{R}_{AB} = \partial_C \hat{\Gamma}^C_{AB} - \partial_B \hat{\Gamma}^C_{AC} + \hat{\Gamma}^C_{AB}\hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{AD}\hat{\Gamma}^D_{BC}, \\ \hat{R} = \hat{g}^{AB}\hat{R}_{AB}, \end{cases} \tag{1.1}$$

where $\hat{g}_{AB}$ is the five-dimensional metric tensor and the capital Latin indices $A, B, \ldots$ run over $0, 1, 2, 3, 4$, Furthermore, the five-dimensional Einstein equations with no five-dimensional energy momentum tensor are defined by $\hat{G}_{AB} = 0$ or equivalently, $\hat{R}_{AB} = 0$, where $\hat{G}_{AB} \equiv \hat{R}_{AB} - \frac{1}{2}\hat{g}_{AB}\hat{R}$. These relations are resulted via variation of a five-dimensional version of the usual Einstein action,

$$\mathcal{S} = -\frac{1}{16\pi\hat{G}} \int \hat{R}\sqrt{-\hat{g}}\; d^4x\; dy \tag{1.2}$$

with respect to the five-dimensional metric, where $\hat{G}$ is a five-dimensional gravitational constant and $y = x^4$ stands for the new fifth coordinate. In general, for the metric, one identifies the $\alpha\beta$-part of $\hat{g}_{AB}$ with the four-dimensional metric tensor $g_{\alpha\beta}$, the $\alpha 4$-part as the electromagnetic potential A_α, and $\hat{g}_{44}$ with a scalar field Φ. Thus the convenient parametrization of it is as follows:

$$\hat{g}_{AB}(x,y) = \begin{pmatrix} g_{\alpha\beta} + \kappa^2\Phi^2 A_\alpha A_\beta & \kappa\Phi^2 A_\alpha \\ & \\ \kappa\Phi^2 A_\beta & \Phi^2 \end{pmatrix}, \tag{1.3}$$

where Greek indices $\alpha, \beta, \ldots$ run over $0, 1, 2, 3$ and κ is a multiplicative factor, which can be denoted by $\kappa = 4\sqrt{\pi G}$ in terms of the four-dimensional gravitational constant. By applying the metric (1.3) and taking into account the cylinder condition, the Einstein action (1.2) consists of the following three components:

$$\mathcal{S} = -\int d^4x\; \sqrt{-g}\; \Phi\left(\frac{R}{16\pi G} + \frac{\Phi^2 F_{\alpha\beta}F^{\alpha\beta}}{4} + \frac{2\;\partial^\alpha\Phi\;\partial_\alpha\Phi}{3\kappa^2\Phi^2}\right), \tag{1.4}$$

where G is described in terms of its (five-dimensional) counterpart $\hat{G}$ by $G \equiv \dfrac{\hat{G}}{\int dy}$. Meanwhile, according to this action, the field equations

$$\delta\mathcal{S} = 0 \longrightarrow \hat{G}_{AB} = 0 \Longleftrightarrow \hat{R}_{AB} = 0 \tag{1.5}$$

reduce to these field equations in terms of four-dimensional quantities [1, 23, 30],

$$\begin{cases} G_{\alpha\beta} = \dfrac{\kappa^2\Phi^2}{2}\; T^{\text{EM}}_{\alpha\beta} - \dfrac{1}{\Phi}\Big[\nabla_\alpha\big(\partial_\beta\Phi\big) - g_{\alpha\beta}\Box\Phi\Big], \\ \nabla^\alpha_{F_{\alpha\beta}} = -3\;\dfrac{\partial^\alpha\Phi}{\Phi}\; F_{\alpha\beta}, \\ \Box\Phi = \dfrac{\kappa^2\Phi^3}{4}\; F_{\alpha\beta}F^{\alpha\beta}, \end{cases} \tag{1.6}$$

where $G_{\alpha\beta} \equiv R_{\alpha\beta} - R\, g_{\alpha\beta}/2$ is the Einstein tensor and $T^{\text{EM}}_{\alpha\beta}$ is the electromagnetic energy-momentum tensor

$$T^{\text{EM}}_{\alpha\beta} \equiv \frac{g_{\alpha\beta}\;F_{\gamma\delta}F^{\gamma\delta}}{4} - F^\gamma_\alpha F_{\beta\gamma}\;, \qquad F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{1.7}$$

Because of the fact that there exists fifteen independent elements in the (five-dimensional) metric (1.3), there are a total of 10+4+1 = 15 equations. Moreover, for Φ = constant, the field equations are just Einstein and Maxwell equations:

$$G_{\alpha\beta} = 8\pi G\Phi^2\, T^{\text{EM}}_{\alpha\beta} \quad , \qquad \nabla^\alpha F_{\alpha\beta} = 0.A_\alpha. \tag{1.8}$$

For more exhaustive details refer to [28]. Since the pioneering research by Van Stockum [32], steady rigid rotation of a fluid in general relativity has been remarkably dealt with many authors. Van Stockum obtained the general metric for an axisymmetric finite rotating dust cloud, and he also proposed the solution for the particular case of an infinite cylinder of rotating dust. The outstanding point in this solution is that with zero pressure the density increased outwards. The investigation of rotating fluids in the context of general relativity received notable consideration principally after Godel [10] proposed relativistic model of a rotating dust universe. Besides, Bonner [4, 5] derived a specific solution from Van Stockum's axisymmetric class in which the dust cloud owned a point singularity at its center and extended to infinity. Moreover, Krasinski [17, 18, 19, 20] and also Herlt [12] have derived rigidly rotating axisymmetric stationary solutions fundamentally by considering specific assumptions regarding four-flow velocities of the perfect fluid. In [7], a comprehensive summary of papers related to the axisymmetric steady rotation of a perfect fluid, incorporating the cylindrically case, is presented. In 1996, Davison [8] reported a one-parameter set of solutions for a fluid admitting the equation of state $p = (2/3)\rho$, rotating about a regular axis. Considering the fact that stationary Kaluza–Klein perfect fluid models in standard Einstein theory are not available in literature, obtaining and analyzing such solutions are so constructive in order to investigate the effects of dimensionality on the different physical parameters. In, Tikekar and Patel [31] have formulated the Kaluza–Klein field equations for cylindrically symmetric rotating distributions of perfect fluid. They have reported a set of physically viable solutions, which is believed to be the first such Kaluza–Klein solutions, and it includes the Kaluza–Klein counterpart of Davidson's solution.

In the following, according to [31], we will present a brief description of Kaluza–Klein field equations for stationary cylindrically symmetric fluid models in standard Einstein theory. For further complete information refer to [31].

A general stationary cylindrically symmetric five-dimensional spacetime is denoted by the following metric:

$$ds^2 = D^2(dt + Hd\phi)^2 - A^2dr^2 - B^2dz^2 - r^2C^2d\phi^2 - E^2d\psi^2, \tag{1.9}$$

where t is the time coordinate, r, z, and ϕ are cylindrical polar coordinates, ψ represents the coordinate corresponding to the extra spatial dimension, and A, B, C, D, and H are functions of the radial coordinate r only. By expressing with respect to pentad

$$\theta^1 = Adr, \quad \theta^2 = Bdz, \quad \theta^3 = rCd\phi, \quad \theta^4 = Ed\psi, \quad \theta^5 = D(dt + Hd\phi), \tag{1.10}$$

the metric (1.9) has the following form:

$$ds^2 = (\theta^5)^2 - (\theta^1)^2 - (\theta^2)^2 - (\theta^3)^2 - (\theta^4)^2. \tag{1.11}$$

If the metric (1.9) denotes the spacetime of a stationary perfect fluid rotating about the regular axis $r = 0$, then the metric coefficients will be related to the dynamical variables through the Einstein field equations, which are in the pentad notation applying the system of units rendering $c = G = 1$, adopted in the form

$$\mathbf{R}_{(ab)} = -8\pi\left[(\rho + p)v_{(a)}v_{(b)} - \frac{1}{3}(\rho - p)g_{(ab)}\right]. \tag{1.12}$$

Here, v_a represents components in the pentad frame of the unit time-like flow vector v^i of the fluid, which satisfies $v^i v_i = 1$. Also, ρ, p denote the matter density and the fluid pressure, respectively. It is more convenient to adapt a coordinate comoving with the observer. Consequently, $v^a = (0, 0, 0, 0, 1)$, and the field equations (1.12) imply the following system of equations:

$$\begin{cases} \mathbf{R}_{(11)} = \mathbf{R}_{(22)} = \mathbf{R}_{(33)} = \mathbf{R}_{(44)} = -\dfrac{8\pi}{3}(\rho - p), \\ \mathbf{R}_{(55)} = -\dfrac{16\pi}{3}(\rho + 2p), \\ \mathbf{R}_{(35)} = 0, \end{cases} \tag{1.13}$$

The field equations comprise a system of six equations relating the two physical parameters ρ and p of the fluid and the six metric coefficients A, B, C, D, E, and H. Significantly, from the first and the second equations of the system (1.13) the following consistency conditions are resulted:

$$\mathbf{R}_{(11)} = \mathbf{R}_{(22)} = \mathbf{R}_{(33)} = \mathbf{R}_{(44)}. \tag{1.14}$$

Davidson [8] obtained a solution of the relativistic system of field equations for a perfect fluid in rigid rotation about a regular axis. His solution suggests the possibility that the system of Kaluza–Klein field equations (1.13) can be solved by assuming the following form for the metric coefficients A, B, C, D, E, and H:

$$\begin{array}{lll} A = (1 + k^2r^2)^a, & B = (1 + k^2r^2)^b, & C = (1 + k^2r^2)^c, \\ D = (1 + k^2r^2)^d, & E = (1 + k^2r^2)^e, & \end{array} \tag{1.15}$$

where a, b, c, d, e, and k are constants. Note that expressions (1.15) confirm the regularity of the metric for all finite r. Equation $\mathbf{R}_{(22)} = \mathbf{R}_{(44)}$ in (1.14) is then satisfied if and only if

$$b = e. \tag{1.16}$$

Accordingly, the equation $\mathbf{R}_{(35)} = 0$ in (1.13) yields the following two significant identities:

$$H = \alpha r^2, \tag{1.17}$$

$$a + c = 2b + 3d, \tag{1.18}$$

where α is the arbitrary constant of integration. Moreover, taking into account relations (1.16) and (1.18), the equation $\mathbf{R}_{(11)} = \mathbf{R}_{(33)}$ contained in (1.14) reduces to the following algebraic relation:

$$2b^2 + 2b(1 + 4d) + d(1 + 2d) = 0. \tag{1.19}$$

Consequently, the Kaluza–Klein field equations are equivalent to the algebraic relations (1.16)–(1.19), relating the seven parameters a, b, c, d, e, α, and k with $H(r)$ as determined by (1.17).

In [31] certain specific cases for physical relevance that follow for certain particular choices of the free parameters, are discussed. In this paper, we will comprehensively analyze the problem of symmetries and conservation laws for the following specific solution, which is reported in [31].

Taking into account the algebraic relations (1.16)–(1.19), the choice $b = e = d = 0$ yields

$$a = -c = -1/2, \qquad \alpha^2 = 2k^2. \tag{1.20}$$

The spacetime of this class of solutions has the following metric:

$$ds^2 = (dt + \sqrt{2}kr^2 d\phi)^2 - \frac{dr^2}{1 + k^2r^2} - dz^2 - r^2(1 + k^2r^2)d\phi^2 - d\psi^2, \tag{1.21}$$

and it characterizes a stationary distribution of fluid with the equation of state $p = \rho$, corresponding to a stiff fluid with uniform density and pressure $\rho = p = \dfrac{k^2}{4\pi}$. For further complete details refer to [31].

This paper is organized as follows: In section 2, we have specifically concentrated on complete investigation of the problem of symmetries for this particular solution mentioned above. First of all, by considering the Lagrangian, which is determined directly from the metric, we will compute the geodesic equations as the Euler–Lagrange equations. Secondly, We obtain the point generators of the one parameter Lie groups of transformations that leave invariant the action integral corresponding to Lagrangian (Noether symmetries). Besides a brief discussion regarding the structure of the corresponding Noether symmetry Lie algebras is presented from the algebraic approach. Section 3 is principally dedicated to the thorough classification of the Noether symmetry subalgebras. Accordingly, it is focused on construction of an optimal system of one-dimensional subalgebras via the adjoint representation. Killing vector fields can be undoubtedly reckoned as one of the most substantial types of symmetries and are denoted by the smooth vector fields, which preserve the metric tensor. Additionally, the flow corresponding to a Killing vector field generates a symmetry in a way that if each point moves on an object at the same distance in the direction of the Killing vector field, then distances on the object will not distorted at all. Therefore, Killing vector fields are inherently expected to be of significant application in the study of geodesic motion. When one investigates the Lagrangian explaining the motion of a particle, one can realize that Killing vectors are the symmetries of the system and lead to conserved canonical momenta analogous to cyclic coordinates in classical mechanics. Taking into account the outstanding properties declared above, section 4 of this paper is particularly devoted to detailed investigation of the Killing vector fields by re-expressing the analyzed metric in the orthogonal frame. Ultimately, in section 5, for the discussed Kaluza–Klein solution, a perfect set of local conservation laws of the system of geodesic equations is calculated. For this purpose, we have applied the celebrated Noether's theore. It is notable that Noether's theorem is fundamentally relied on the geodesic Lagrangian and the corresponding Noether symmetries, which leave the action integral invariant. Meanwhile, some concluding remarks are declared at the end of the paper.

2. Noether symmetries

In 1918, Noether [25] presented her outstanding theorem to obtain local conservation laws for systems of differential equations, which admit a variational principle. Indeed, she demonstrated that if a system of differential equations admits a variational principle (action integral), then any one-parameter Lie group of point transformations that leaves invariant the action functional results a local conservation law. Specifically, she proposed an explicit formula for the fluxes of the conservation law. The noticeable fact is that when an arbitrary system of differential equations admits a variational principle, then the extremals of its action functional yield the Euler–Lagrange equations. In this case, taking into account Noether's celebrated theorem, if one has a point symmetry of the action integral, then the fluxes of a local conservation law are explicitly determined via a formula, which comprises the infinitesimal of the point symmetry and the corresponding Lagrangian density of the action integral. In the following, Noether's formulation of her theorem is presented [3, 2]. According to this formulation, it is implicated that the action integral $J[U]$ to be invariant under the following one-parameter Lie group of point transformations:

$$\begin{cases} \big(x^*\big)^i = x^i + \varepsilon\xi^i\big(x,U\big) + O\big(\varepsilon^2\big), & i = 1,\dots,n, \\ \big(U^*\big)^\mu = U^\mu + \varepsilon\Omega^\mu\big(x,U\big) + O\big(\varepsilon^2\big), & \mu = 1,\dots,m. \end{cases} \tag{2.1}$$

with associated infinitesimal generator defined by

$$X = \xi^i\big(x,U\big)\frac{\partial}{\partial x^i} + \Omega^\nu\big(x,U\big)\frac{\partial}{\partial U^\nu}. \tag{2.2}$$

Suppose that a functional $J[U]$ is defined on a domain Δ in terms of n independent variables $x = \big(x^1,\dots,x^n\big)$ and m arbitrary functions $U = \big(U^1(x),\dots,U^m(x)\big)$ and that their corresponding partial derivatives to order k are as follows:

$$J(U) = \int_\Delta L\big[U\big]\ dx = \int_\Delta L\Big(x,U,\partial U,\dots,\partial^k U\Big)\ dx. \tag{2.3}$$

The function $L[U] = L\big(x,U,\partial U,\dots,\partial^k U\big)$ is called a Lagrangian, and the functional $J[U]$ is denoted by an action integral. Therefore, invariance hold if and only if $\displaystyle\int_{\Delta^*} L\big[U^*\big]dx^* = \int_\Delta L\big[U\big]dx$, where Δ^* is the image of Δ under the point transformation (2.1). Furthermore, the Jacobian of the transformation (2.1) is characterized as follows:

$$\mathrm{J} = \det\Big(D_i\big(x^*\big)^j\Big) = 1 + \varepsilon D_i\xi^i\big(x,U\big) + O\big(\varepsilon^2\big). \tag{2.4}$$

Hence, $dx^* = \mathrm{J}dx$. Furthermore, considering the point that (2.1) is a Lie group of transformations, in terms of the kth prolongation of the infinitesimal generator (2.2), the following parity $L\big[U^*\big] = \exp\big(\varepsilon X^{(k)}\big)L\big[U\big]$ is deduced. Accordingly, in Noether's formulation, the one-parameter Lie group of point transformations

(2.1) is a point symmetry of $J[U]$ (2.3) if and only if for arbitrary $U(x)$ the following identity holds [3]:

$$\begin{aligned}\int_{\Delta}\Big(\mathrm{J}\exp\big(\varepsilon X^{(k)}\big)-1\Big)L\big[U\big]dx \\ =\varepsilon\int_{\Delta}\Big(L\big[U\big]\Big(D_i\xi^i\big(x,U\big)\Big)+X^{(k)}L\big[U\big]\Big)dx+O\big(\varepsilon^2\big),\end{aligned}\tag{2.5}$$

where $X^{(k)}$ is the kth prolongation of the infinitesimal generator (2.2) expressed by

$$\begin{aligned}X^{(k)}=\xi^i(x,U)\frac{\partial}{\partial x^i}+\Omega^\mu(x,U)\frac{\partial}{\partial U^\mu}+\Omega_i^{(1)\mu}\big(x,U,\partial U\big)\frac{\partial}{\partial U_i^\mu} \\ +\cdots+\Omega^{(k)\mu}_{i_1\dots i_k}\big(x,U,\partial U,\dots,\partial^k U\big)\frac{\partial}{\partial U^\mu_{i_1\dots i_k}}.\end{aligned}\tag{2.6}$$

Moreover, the extended infinitesimals are defined as follows:

$$\begin{cases}\Omega_i^{(1)\mu}=D_i\Omega^\mu-\big(D_i\xi^j\big)U_j^\mu, \\ \Omega^{(k)\mu}_{i_1\dots i_k}=D_{i_k}\Omega^{(k-1)\mu}_{i_1\dots i_{k-1}}-\big(D_{i_k}\xi^j\big)U^\mu_{i_1\dots i_{k-1}j}, \\ \mu=1,\dots,m,\quad i,i_j=1,\dots,n\ \text{ for }\ j=1,\dots,k\ \text{ with }\ k=2,3,\dots.\end{cases}\tag{2.7}$$

Consequently, if $J[U]$ (2.3) possesses the point symmetry (2.2), then the $O(\varepsilon^2)$ term in (2.5) vanishes. As a consequence, the following significant identity is resulted [3]:

$$L[U]D_i\xi^i(x,U)+X^{(k)}L[U]\equiv 0.\tag{2.8}$$

It is noticeable that variational symmetries are of particular importance principally considering the fact that due to celebrated Noether's theorem [25] there is a procedure that connects the constants of the motion of an arbitrary Lagrangian system to its corresponding symmetry transformations [13, 15]. In addition, it is worth mentioning that the geodesic equation can be also derived via the action principle. Since the associated Euler–Lagrange equations (geodesic equation) are second order ordinary differential equations, one generally takes first order Lagrangians. Specifically, we consider $L(s,x^\mu,\dot{x}^\mu)$, where "$\cdot$" denotes differentiation with respect to the arc length parameter s, for minimizing the arc length (written from the square of the arc length for convenience). Hence, the action will be defined as

$$J=\frac{1}{2}\int g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}d\tau.\tag{2.9}$$

Now, by varying this action with respect to the curve x^μ and through straightforward application of the Euler–Lagrange equation $\frac{d}{d\tau}\frac{\partial L}{\partial\dot{x}^\mu}-\frac{\partial L}{\partial x^\mu}=0$ with Lagrangian $L[x^\mu,\dot{x}^\mu]=\frac{1}{2}g_{\mu\nu}(x^\kappa)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds}$, the following equivalent formulation of the geodesic equation is determined:

$$\frac{d^2x^\sigma}{d\tau^2}+\Gamma^\sigma_{\rho\nu}\frac{dx^\rho}{d\tau}\frac{dx^\nu}{d\tau}=0,\quad \Gamma^\sigma_{\rho\nu}=\frac{1}{2}g^{\sigma\mu}\Big[\partial_\rho g_{\mu\nu}+\partial_\nu g_{\mu\rho}-\partial_\mu g_{\rho\nu}\Big].\tag{2.10}$$

Consequently, as discussed above, according to Noether's formulation of Noether's theorem, Noether symmetries, or symmetries of a Lagrangian are defined as follows: A variational integral defined as $I = \int_V L(s, x^\mu, \dot{x}^\mu)\ ds$ is called to be invariant under a one parameter Lie group of transformations:

$$\begin{cases} s^* = s^*(s, \mathbf{x}; \varepsilon) = s + \varepsilon\xi(s, \mathbf{x}) + O(\varepsilon^2), \\ x^{\mu *} = x^{\mu *}(s, \mathbf{x}; \varepsilon) = x^\mu + \varepsilon\Omega^\mu(s, \mathbf{x}) + O(\varepsilon^2) \end{cases} \tag{2.11}$$

with associated infinitesimal generator given by

$$\mathbf{X} = \xi(s, x^\mu)\frac{\partial}{\partial s} + \Omega^\nu(s, x^\mu)\frac{\partial}{\partial x^\nu}, \tag{2.12}$$

if the following identity holds: $\int_{V^*} L(s^*, x^{\mu *}, \dot{x}^{\mu *})\ ds^* = \int_V L(s, x^\mu, \dot{x}^\mu)\ ds$. Meanwhile, it is noticeable that V^* is a volume obtained from V under the considered transformation (2.11). In other words, the invariance of $I = \int_V L(s, x^\mu, \dot{x}^\mu)\ ds$ up to gauge $A = A(s, x^\mu)$ can be thoroughly described by

$$\int_{V^*} L(s^*, x^{\mu *}, \dot{x}^{\mu *})\ ds^* = \int_V \left[L(s, x^\mu, \dot{x}^\mu) + \varepsilon\frac{dA}{ds}\right]\ ds. \tag{2.13}$$

Now, taking into account $\frac{d}{ds} = \frac{ds^*}{ds}\frac{d}{ds^*}$ and differentiating (2.13) with respect to s, we have

$$\big(1 + \varepsilon\frac{d\xi}{ds}\big)L(s^*, x^{\mu *}, \dot{x}^{\mu *}) = L(s, x^\mu, \dot{x}^\mu) + \varepsilon\frac{dA}{ds}. \tag{2.14}$$

Now, considering the one-parameter Lie group of point transformations (2.11), the following identities are resulted:

$$\begin{aligned} &\big(1 + \varepsilon\frac{d\xi}{ds}\big)L(s + \varepsilon\xi, x^\mu + \varepsilon\Omega^\mu, \dot{x}^\mu + \varepsilon\Omega^\mu_{,s}) = L(s, x^\mu, \dot{x}^\mu) + \varepsilon\frac{dA}{ds}, \\ &\big(1 + \varepsilon\frac{d\xi}{ds}\big)\left[L(s, x^\mu, \dot{x}^\mu) + \varepsilon\frac{\partial L}{\partial s}\xi + \varepsilon\frac{\partial L}{\partial x^\mu}\Omega^\mu + \varepsilon\frac{\partial L}{\partial \dot{x}^\mu}\Omega^\mu_{,s}\right] = L(s, x^\mu, \dot{x}^\mu) + \varepsilon\frac{dA}{ds}, \\ &\varepsilon\frac{\partial L}{\partial s}\xi + \varepsilon\frac{\partial L}{\partial x^\mu}\Omega^\mu + \varepsilon\frac{\partial L}{\partial \dot{x}^\mu}\Omega^\mu_{,s} + \varepsilon\frac{d\xi}{ds}L(s, x^\mu, \dot{x}^\mu) + O(\varepsilon^2) = \varepsilon\frac{dA}{ds}. \end{aligned} \tag{2.15}$$

Overall, taking into account the identity (2.15) and by neglecting $O(\varepsilon^2)$, the following relation is obtained:

$$\mathbf{X}^{(1)}L + (D_s\xi)L = D_sA, \tag{2.16}$$

where $D_s = \frac{\partial}{\partial s} + \dot{x}^\mu\frac{\partial}{\partial x^\mu}$, which is defined on the real parameter fiber bundle over the tangent bundle to the manifold and $\mathbf{X}^{(1)}$ is the first prolongation of the vector field (2.12) and is expressed by

$$\mathbf{X}^{(1)} = \mathbf{X} + \big(\Omega^\nu_{,s} + \Omega^\nu_{,\mu}\dot{x}^\mu - \xi_{,s}\dot{x}^\nu - \xi_{,\mu}\dot{x}^\mu\dot{x}^\nu\big)\frac{\partial}{\partial \dot{x}^\nu}. \tag{2.17}$$

Then $\mathbf{X}$ is called a *Noether point symmetry* of this Lagrangian (2.9).

Remark 2.1. Taking into account the fact that analysis of the problem of symmetries and conservation laws for some specific solutions of Kaluza–Klein field equations for stationary symmetric fluid models in standard Einstein theory is the main concentration of the current research, throughout the paper we specifically deal with stationary cylindrically symmetric five-dimensional spacetimes. In other words, in the metric, which is discussed here, t is the time coordinate, r, z, and ϕ are cylindrical polar coordinates, and ψ represents the coordinate corresponding to the extra spatial dimension. Consequently, in above relations we have $\mu, \nu = 1, 2, 3, 4, 5$.

In this section, first of all, by considering the Lagrangian that is determined directly from the metric (1.21), we will compute the geodesic equations as the Euler Lagrange equations. Secondly, We obtain the point generators of the one parameter Lie groups of transformations that leave invariant the action integral corresponding to the Lagrangian (Noether symmetries).

2.1. **Computation of the Noether symmetries for solution (1.21).** The Lagrangian for the metric (1.21) is

$$L = \dot{t}^2 - \frac{\dot{r}^2}{1+k^2r^2} - \dot{z}^2 + (k^2r^4 - r^2)\dot{\phi}^2 + 2\sqrt{2}kr^2\dot{t}\dot{\phi} - \dot{\psi}^2. \tag{2.18}$$

The corresponding simplified Euler–Lagrange equations are the geodesic equations given by

$$\mathbf{E^{(I)}} : \begin{cases} \mathbf{E_1} : \ddot{t} + \dfrac{16rk^2}{7k^2r^2+1}\,\dot{t}\dot{r} + \dfrac{4\sqrt{2}k^3r^3}{7k^2r^2+1}\,\dot{r}\dot{\phi} = 0, \\ \mathbf{E_2} : \ddot{r} + 4\sqrt{2}kr(1+k^2r^2)\,\dot{t}\dot{\phi} - \dfrac{rk^2}{1+k^2r^2}\,\dot{r}^2 \\ \qquad +r(1+k^2r^2)(2k^2r^2-1)\,\dot{\phi}^2 = 0, \\ \mathbf{E_3} : \ddot{z} = 0, \\ \mathbf{E_4} : \ddot{\phi} - \dfrac{4\sqrt{2}k}{r(7k^2r^2+1)}\,\dot{t}\dot{r} + \dfrac{2(6k^2r^2+1)}{r(7k^2r^2+1)}\,\dot{r}\dot{\phi} = 0, \\ \mathbf{E_5} : \ddot{\psi} = 0. \end{cases} \tag{2.19}$$

By applying (2.18) in (2.16), we obtain the determining (partial differential) equations for seven unknown functions ξ, Ω^μ, and A, where each of these is a function of six variables, that is, s, t, r, z, ϕ, and ψ. Solving these equations for the metric (1.21), the following set of solutions is concluded.

Theorem 2.2. *The Lie group of Noether symmetries corresponding to solution* (1.21) *has a Lie algebra generated by the vector fields* $\mathbf{X} = \xi\frac{\partial}{\partial s} + \Omega^1\frac{\partial}{\partial t} + \Omega^2\frac{\partial}{\partial r} +$

$\Omega^3\frac{\partial}{\partial z}+\Omega^4\frac{\partial}{\partial \phi}+\Omega^5\frac{\partial}{\partial \psi}$, *where*

$$\begin{aligned}
&\xi(s,t,r,z,\phi,\psi)=c_1,\\
&\Omega^1(s,t,r,z,\phi,\psi)=\frac{\sqrt{2}rk}{\sqrt{1+k^2r^2}}\big(c_8\cos\phi-c_9\sin\phi\big)+c_{10},\\
&\Omega^2(s,t,r,z,\phi,\psi)=\sqrt{1+k^2r^2}\big(c_8\sin\phi+c_9\cos\phi\big),\\
&\Omega^3(s,t,r,z,\phi,\psi)=-\frac{1}{2}c_4s+c_5\psi+c_6,\\
&\Omega^4(s,t,r,z,\phi,\psi)=\frac{2k^2r^2+1}{r\sqrt{1+k^2r^2}}\big(c_8\cos\phi-c_9\sin\phi\big)+c_{11},\\
&\Omega^5(s,t,r,z,\phi,\psi)=-\frac{1}{2}c_2s-c_5z+c_7,\\
&A(s,t,r,z,\phi,\psi)=c_4z+c_2\psi+c_3.
\end{aligned}\tag{2.20}$$

and $c_i,\ i=1,\dots,11$ *are arbitrary constants.*

From (2.20), we obtain the ten-dimensional Lie algebra of Noether point symmetries with the following basis.

Corollary 2.3. *Infinitesimal generators of every one parameter Lie group of Noether symmetries associated to* (1.21) *are as follows:*

$$\begin{aligned}
&\mathbf{X_1}=\frac{\partial}{\partial s}\quad A=c,\quad \mathbf{X_2}=\frac{\partial}{\partial t}\quad A=c,\quad \mathbf{X_3}=\frac{\partial}{\partial z}\qquad A=c,\\
&\mathbf{X_4}=\frac{\partial}{\partial \phi}\quad A=c,\quad \mathbf{X_5}=\frac{\partial}{\partial \psi}\quad A=c,\quad \mathbf{X_6}=-\frac{1}{2}s\frac{\partial}{\partial z}\quad A=z+c,\\
&\mathbf{X_7}=-\frac{1}{2}s\frac{\partial}{\partial \psi}\quad A=\psi+c,\quad \mathbf{X_8}=-\psi\frac{\partial}{\partial z}+z\frac{\partial}{\partial \psi}\quad A=c,\\
&\mathbf{X_9}=\frac{\sqrt{2}rk\cos\phi}{\sqrt{1+k^2r^2}}\ \frac{\partial}{\partial t}+\sin\phi\sqrt{1+k^2r^2}\ \frac{\partial}{\partial r}+\frac{(2k^2r^2+1)\cos\phi}{r\sqrt{1+k^2r^2}}\ \frac{\partial}{\partial \phi}\quad A=c,\\
&\mathbf{X_{10}}=-\frac{\sqrt{2}rk\sin\phi}{\sqrt{1+k^2r^2}}\ \frac{\partial}{\partial t}+\cos\phi\sqrt{1+k^2r^2}\ \frac{\partial}{\partial r}-\frac{(2k^2r^2+1)\sin\phi}{r\sqrt{1+k^2r^2}}\ \frac{\partial}{\partial \phi}\quad A=c.
\end{aligned}\tag{2.21}$$

The commutator table of Noether symmetry generators of the system of geodesic equations (2.19) *is given in Table 1, where the entry in the* i*th row and* j*th column is defined as* $[X_i,X_j]=X_iX_j-X_jX_i,\quad i,j=1,\dots,10.$

Let $\mathfrak{g}^{\mathrm{I}}$ denote the Lie algebra of local symmetries corresponding to the system of geodesic equations (2.19). In this section, a brief discussion regarding the algebraic structure of $\mathfrak{g}^{\mathrm{I}}$ is presented. The Lie algebra $\mathfrak{g}^{\mathrm{I}}$ is nonsolvable, because

TABLE 1. Commutation relations satisfied by infinitesimal generators of $\mathfrak{g}^{\mathrm{I}}$

[,]	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$	$\mathbf{X_4}$	$\mathbf{X_5}$	$\mathbf{X_6}$	$\mathbf{X_7}$	$\mathbf{X_8}$	$\mathbf{X_9}$	$\mathbf{X_{10}}$
$\mathbf{X_1}$	0	0	0	0	0	$-\frac{1}{2}\mathbf{X_3}$	$-\frac{1}{2}\mathbf{X_5}$	0	0	0
$\mathbf{X_2}$	0	0	0	0	0	0	0	0	0	0
$\mathbf{X_3}$	0	0	0	0	0	0	0	$\mathbf{X_5}$	0	0
$\mathbf{X_4}$	0	0	0	0	0	0	0	0	$\mathbf{X_{10}}$	$-\mathbf{X_9}$
$\mathbf{X_5}$	0	0	0	0	0	0	0	$-\mathbf{X_3}$	0	0
$\mathbf{X_6}$	$\frac{1}{2}\mathbf{X_3}$	0	0	0	0	0	0	$\mathbf{X_7}$	0	0
$\mathbf{X_7}$	$\frac{1}{2}\mathbf{X_5}$	0	0	0	0	0	0	$-\mathbf{X_6}$	0	0
$\mathbf{X_8}$	0	0	$-\mathbf{X_5}$	0	$\mathbf{X_3}$	$-\mathbf{X_7}$	$\mathbf{X_6}$	0	0	0
$\mathbf{X_9}$	0	0	0	$-\mathbf{X_{10}}$	0	0	0	0	0	$-4k^2\mathbf{X_4}-2\sqrt{2}k\mathbf{X_2}$
$\mathbf{X_{10}}$	0	0	0	$\mathbf{X_9}$	0	0	0	0	$4k^2\mathbf{X_4}+2\sqrt{2}k\mathbf{X_2}$	0

if $\mathfrak{g}^{\mathrm{I}(1)} =< X_i, [X_i, X_j] >= [\mathfrak{g}^{\mathrm{I}}, \mathfrak{g}^{\mathrm{I}}]$ be the derived subalgebra of $\mathfrak{g}^{\mathrm{I}}$, then we have

$$\begin{aligned}
\mathfrak{g}^{\mathrm{I}(1)} &= [\mathfrak{g}^{\mathrm{I}}, \mathfrak{g}^{\mathrm{I}}] =< -\frac{1}{2}X_3, -\big(2\sqrt{2}kX_2 + 4k^2X_4\big), -\frac{1}{2}X_5, -X_6, X_7, -X_9, X_{10} >,\\
\mathfrak{g}^{\mathrm{I}(2)} &= [\mathfrak{g}^{\mathrm{I}(1)}, \mathfrak{g}^{\mathrm{I}(1)}] =< -\big(2\sqrt{2}kX_2 + 4k^2X_4\big), -4k^2X_9, -4k^2X_{10} >\\
&= [\mathfrak{g}^{\mathrm{I}(2)}, \mathfrak{g}^{\mathrm{I}(2)}] = \mathfrak{g}^{\mathrm{I}(3)}.
\end{aligned}$$

Thus, we have the following chain of ideals $\mathfrak{g}^{\mathrm{I}} \supset \mathfrak{g}^{\mathrm{I}(1)} \supset \mathfrak{g}^{\mathrm{I}(2)} = \mathfrak{g}^{\mathrm{I}(3)} \neq 0$, which shows the nonsolvability of $\mathfrak{g}^{\mathrm{I}}$. Also, $\mathfrak{g}^{\mathrm{I}}$ is not semisimple, because its killing form

$$\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&-2&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-4&0&0\\
0&0&0&0&0&0&0&0&8k^2&0\\
0&0&0&0&0&0&0& &0&8k^2
\end{pmatrix}$$

is degenerate. Moreover, $\mathfrak{g}^{\mathrm{I}}$ has a Levi decomposition of the form $\mathfrak{g}^{\mathrm{I}} = \mathfrak{r} \ltimes \mathfrak{h}$, where $\mathfrak{r} =< X_1, X_2, X_3, X_5, X_6, X_7, X_8 >$ is the radical (the largest solvable ideal) of $\mathfrak{g}^{\mathrm{I}}$ and $\mathfrak{h} =< X_2 + \sqrt{2}kX_4, X_9, X_{10} >$ is a semisimple and nonsolvable subalgebra of $\mathfrak{g}^{\mathrm{I}}$. Hence, the quotient algebra generated from $\mathfrak{g}^{\mathrm{I}}$ can be obtained such as

$$\mathfrak{g}^{\mathrm{I}}_1 = \mathfrak{g}^{\mathrm{I}}/\mathfrak{r} = \big\{X + \mathfrak{r} \mid X \in \mathfrak{g}^{\mathrm{I}}\big\}. \tag{2.22}$$

The members of $\mathfrak{g}_1^{\mathrm{I}}$ are denoted by Y_i, and the commutator table of the resulted quotient Lie algebra is given in Table 2, where the entry in the ith row and jth column is defined as $[Y_i, Y_j] = Y_iY_j - Y_jY_i,\ i, j = 1, 2, 3.$

TABLE 2. Commutation table of $\mathfrak{g}_1^{\mathrm{I}}$

[,]	Y_1	Y_2	Y_3
Y_1	0	$\sqrt{\mathbf{2}}\ \mathbf{k}\ \mathbf{Y_3}$	$-\sqrt{\mathbf{2}}\ \mathbf{k}\ \mathbf{Y_2}$
Y_2	$-\sqrt{\mathbf{2}}\ \mathbf{k}\ \mathbf{Y_3}$	0	$-\mathbf{2}\sqrt{\mathbf{2}}\ \mathbf{k}\ \mathbf{Y_1}$
Y_3	$-\sqrt{\mathbf{2}}\ \mathbf{k}\ \mathbf{Y_2}$	$\mathbf{2}\sqrt{\mathbf{2}}\ \mathbf{k}\ \mathbf{Y_1}$	0

The relation (2.22) has noteworthy consequences on the reduction of geodesic equations (2.19), which is not the purpose of this paper, but we will present a brief description regarding this issue in the following.

Indeed, for the integration of an involutive distribution, the process decomposes into the two following steps [21]:

(i) : integration of the involutive distribution with symmetry Lie algebra $\mathfrak{g}/\mathfrak{r}$, and

(ii) : integration on integral manifolds with symmetry algebra $\mathfrak{r}$.

First, applying this procedure to the radical $\mathfrak{r}$, the integration problem would be decomposed into the following two parts: the integration of the distribution with semisimple algebra $\mathfrak{g}/\mathfrak{r}$ and then integration of the distribution, which is restricted to the integral manifold with the solvable symmetry algebra $\mathfrak{r}$.

The last step can be accomplished via quadratures. Moreover, every semisimple Lie algebra $\mathfrak{g}/\mathfrak{r}$ is a direct sum of simple ones, which are ideal in $\mathfrak{g}/\mathfrak{r}$. Hence, according to the Lie–Bianchi theorem, the integration problem is reduced to involutive distributions equipped with simple algebras of symmetries (refer to [21] for complete details).

The quotient algebra $\mathfrak{g}_1^{\mathrm{I}}$ is a semisimple and nonsolvable Lie algebra. It is semsimple, because its killing form

$$\begin{pmatrix} -4k^2 & 0 & 0 \\ 0 & 8k^2 & 0 \\ 0 & 0 & 8k^2 \end{pmatrix}$$

is nondegenerate. Moreover, $\mathfrak{g}_1^{\mathrm{I}}$ is nonsolvable, because if $\mathfrak{g}_1^{\mathrm{I}(1)} =< Y_i, [Y_i, Y_j] >= [\mathfrak{g}_1^{\mathrm{I}}, \mathfrak{g}_1^{\mathrm{I}}]$ be the derived subalgebra of $\mathfrak{g}_1^{\mathrm{I}}$, we have:

$$\mathfrak{g}_1^{\mathrm{I}} = \mathfrak{g}_1^{\mathrm{I}(1)} = [\mathfrak{g}_1^{\mathrm{I}}, \mathfrak{g}_1^{\mathrm{I}}] =< Y_1, Y_2, Y_3 > .$$

So, it is inferred that $\mathfrak{g}_1^{\mathrm{I}} = \mathfrak{g}_1^{\mathrm{I}(1)} \neq 0$, which shows the nonsolvability of $\mathfrak{g}_1^{\mathrm{I}}$.

3. Classification of Noether symmetry subalgebras for the system of geodesic equations

Let G denote the Lie group of Noether symmetries of the system of geodesic equations. Now, G operates on the set of the solutions of equation denoted by Θ.

Let H be an r-dimensional subgroup of G and let $s \cdot G$ be the orbit of s. Then, H-invariant solutions $s \in \Theta$ are determined by equality $s \cdot \Theta = \{s\}$. If $h \in G$ is a transformation and $s \in \Theta$, then $h \cdot (s \cdot H) = (h \cdot s) \cdot (hHh^{-1})$. So according to [26] it is deduced that every H-invariant solution s transforms into an hHh^{-1}-invariant solution. Therefore, from similar subgroups of G, different invariant solutions are obtained. Hence, the classification of H-invariant solutions is reduced to the problem of classification of subgroups of G, up to similarity. A list of conjugacy inequivalent r-dimensional subgroups of G, with the property that any other subgroup is conjugate to precisely one subgroup in the list, is called an optimal system of r-dimensional subgroups of G. Similarly, if every member of a list of r-dimensional subalgebras of $\mathfrak{g}$ is equivalent to a unique member of the list under some element of the adjoint representation, $\tilde{\mathfrak{h}} = \mathrm{Ad}(g) \cdot \mathfrak{h}$, $g \in G$, then an optimal system of r-dimensional subalgebras is generated. Let H and $\tilde{H}$ be two connected, r-dimensional Lie subgroups of the Lie group G with corresponding Lie subalgebras $\mathfrak{h}$ and $\tilde{\mathfrak{h}}$ of the Lie algebra $\mathfrak{g}$ of G. Then according to [26, 29], $\tilde{H} = gHg^{-1}$ are conjugate subgroups if and only if $\tilde{\mathfrak{h}} = \mathrm{Ad}(g) \cdot \mathfrak{h}$ are conjugate subalgebras. Subsequently, the problem of determining an optimal system of subgroups is equivalent to that of finding an optimal system of subalgebras. So we will focus on this issue in the following. There is obviously an infinite number of one-dimensional subalgebras of the system of geodesic equation Lie algebra, each of which correspond a family of group invariant solutions. Hence, applying all the one-dimensional subalgebras of for constructing the invariant solutions is impossible. Indeed there is an effective, systematic means of classifying these solutions, leading to an optimal system of group-invariant solutions from which every other such solution can be derived [26, 27]. This procedure involves constructing the adjoint representation group that introduces a conjugate relation in the set of all one-dimensional subalgebras. Taking into account the fact that each one-dimensional subalgebra is characterized via a nonzero vector in the corresponding symmetry Lie algebra. For one-dimensional subalgebras, this problem is essentially the same as the problem of classifying the orbits of the adjoint representation. An optimal set of subalgebras is generated, whenever we select only one representative from each family of equivalent subalgebras. The corresponding set of invariant solutions is then the minimal list from which we can obtain all other invariant solutions of one-dimensional subalgebras simply via transformations. Meanwhile, each X_i of the basis infinitesimal symmetries generates an adjoint representation (or interior automorphism) $\mathrm{Ad}(\exp(\varepsilon X_i))$ defined by the Lie series as follows:

$$\mathrm{Ad}(\exp(\varepsilon.X_i).X_j) = X_j - \varepsilon.[X_i, X_j] + \frac{\varepsilon^2}{2}.[X_i, [X_i, X_j]] - \cdots , \tag{3.1}$$

where $[X_i, X_j]$ is the commutator for the Lie algebra, ε is a parameter. In the following, the Noether symmetries associated to the system of geodesic equations (2.19) are thoroughly classified via the adjoint representation and an optimal system of one-dimensional subalgebras, which provides the preliminary classification of group invariant solutions for the system of geodesic equations is constructed.

3.1. **Classification of Noether symmetries via the adjoint representation for solution (1.21).** We can expect to simplify a given arbitrary element,

$$\mathbf{X} = a_1\mathbf{X_1} + a_2\mathbf{X_2} + \cdots + a_{10}\mathbf{X_{10}} \tag{3.2}$$

of the Lie algebra of Noether symmetries associated to the geodesic Lagrangian (2.18), which was denoted by $\mathfrak{g}^{\mathrm{I}}$. Note that the elements of $\mathfrak{g}^{\mathrm{I}}$ can be represented by vectors $a = (a_1, \ldots, a_{10}) \in \mathbb{R}^{10}$ since each of them can be written in the form (3.2) for some constants $a_1, \ldots, a_{10}$. Hence, the adjoint action can be regarded as (in fact is) a group of linear transformations of the vectors $(a_1, \ldots, a_{10})$.

Therefore, we can state the following theorem.

Theorem 3.1. *An optimal system of one-dimensional Lie subalgebras of Noether symmetries associated to the geodesic Lagrangian* (2.18) *is provided by those generated by*

$$\begin{aligned}
(\mathbf{1}) : \ & a_1\mathbf{X_1} + a_2\mathbf{X_2} + a_4\mathbf{X_4} + \mathbf{X_5} + a_6\mathbf{X_6} + a_{10}\mathbf{X_{10}} \\
&= a_1\frac{\partial}{\partial s} + \left(a_2 - a_{10}\frac{\sqrt{2}rk\sin\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t} + \left(a_{10}\cos\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r} - a_6\left(\frac{s}{2}\right)\frac{\partial}{\partial z} \\
&\quad + \left(a_4 - a_{10}\frac{\left(2k^2r^2+1\right)\sin\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\psi}, \qquad A = z + c.
\end{aligned}$$

$$\begin{aligned}
(\mathbf{2}) : \ & a_1\mathbf{X_1} + a_2\mathbf{X_2} + a_4\mathbf{X_4} + \mathbf{X_5} + a_6\mathbf{X_6} + a_9\mathbf{X_9} \\
&= a_1\frac{\partial}{\partial s} + \left(a_2 + a_9\frac{\sqrt{2}rk\cos\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t} + \left(a_9\sin\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r} - a_6\left(\frac{s}{2}\right)\frac{\partial}{\partial z} \\
&\quad + \left(a_4 + a_9\frac{\left(2k^2r^2+1\right)\cos\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi} + \frac{\partial}{\partial\psi}, \qquad A = z + c.
\end{aligned}$$

$$\begin{aligned}
(\mathbf{3}) : \ & a_1\mathbf{X_1} + a_2\mathbf{X_2} + \mathbf{X_3} + a_4\mathbf{X_4} + a_7\mathbf{X_7} + a_9\mathbf{X_9} \\
&= a_1\frac{\partial}{\partial s} + \left(a_2 + a_9\frac{\sqrt{2}rk\cos\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t} + \left(a_9\sin\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r} + \frac{\partial}{\partial z} \\
&\quad + \left(a_4 + a_9\frac{\left(2k^2r^2+1\right)\cos\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi} - \left(a_7\frac{s}{2}\right)\frac{\partial}{\partial\psi}, \qquad A = \psi + c.
\end{aligned}$$

$$\begin{aligned}
(\mathbf{4}) : \ & a_1\mathbf{X_1} + a_2\mathbf{X_2} + \mathbf{X_3} + a_4\mathbf{X_4} + a_7\mathbf{X_7} + a_{10}\mathbf{X_{10}} \\
&= a_1\frac{\partial}{\partial s} + \left(a_2 - a_{10}\frac{\sqrt{2}rk\sin\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t} + \left(a_{10}\cos\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r} + \frac{\partial}{\partial z} \\
&\quad + \left(a_4 - a_{10}\frac{\left(2k^2r^2+1\right)\sin\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi} - \left(a_7\frac{s}{2}\right)\frac{\partial}{\partial\psi}, \qquad A = \psi + c.
\end{aligned}$$

$$(\mathbf{5}):\ a_1\mathbf{X_1}+a_2\mathbf{X_2}+a_4\mathbf{X_4}+\mathbf{X_7}+a_9\mathbf{X_9}$$
$$=a_1\frac{\partial}{\partial s}+\left(a_2+a_9\frac{\sqrt{2}rk\cos\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t}+\left(a_9\sin\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r}+\frac{\partial}{\partial z}$$
$$+\left(a_4+a_9\frac{\left(2k^2r^2+1\right)\cos\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi}-\left(\frac{s}{2}\right)\frac{\partial}{\partial\psi},\qquad A=\psi+c.$$

$$(\mathbf{6}):\ a_1\mathbf{X_1}+a_2\mathbf{X_2}+a_4\mathbf{X_4}+\mathbf{X_7}+a_{10}\mathbf{X_{10}}$$
$$=a_1\frac{\partial}{\partial s}+\left(a_2-a_{10}\frac{\sqrt{2}rk\sin\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t}+\left(a_{10}\cos\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r}+\frac{\partial}{\partial z}$$
$$+\left(a_4-a_{10}\frac{\left(2k^2r^2+1\right)\sin\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi}-\left(\frac{s}{2}\right)\frac{\partial}{\partial\psi},\qquad A=\psi+c.$$

$$(\mathbf{7}):\ a_1\mathbf{X_1}+a_2\mathbf{X_2}+a_4\mathbf{X_4}+\mathbf{X_6}+a_{10}\mathbf{X_{10}}$$
$$=a_1\frac{\partial}{\partial s}+\left(a_2-a_{10}\frac{\sqrt{2}rk\sin\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t}+\left(a_{10}\cos\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r}$$
$$-\left(\frac{s}{2}\right)\frac{\partial}{\partial z}+\left(a_4-a_{10}\frac{\left(2k^2r^2+1\right)\sin\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi}+\frac{\partial}{\partial\psi},\quad A=z+c.$$

$$(\mathbf{8}):\ a_1\mathbf{X_1}+a_2\mathbf{X_2}+a_4\mathbf{X_4}+\mathbf{X_6}+a_9\mathbf{X_9}$$
$$=a_1\frac{\partial}{\partial s}+\left(a_2+a_9\frac{\sqrt{2}rk\cos\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t}+\left(a_9\sin\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r}$$
$$-\left(\frac{s}{2}\right)\frac{\partial}{\partial z}+\left(a_4+a_9\frac{\left(2k^2r^2+1\right)\cos\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi}+\frac{\partial}{\partial\psi},\quad A=z+c.$$

$$(\mathbf{9}):\ a_1\mathbf{X_1}+a_2\mathbf{X_2}+a_4\mathbf{X_4}+a_8\mathbf{X_8}+\mathbf{X_{10}}$$
$$=a_1\frac{\partial}{\partial s}+\left(a_2-\frac{\sqrt{2}rk\sin\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t}+\left(\cos\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r}-\left(a_8\psi\right)\frac{\partial}{\partial z}$$
$$+\left(a_4-\frac{\left(2k^2r^2+1\right)\sin\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi}+\left(a_8z\right)\frac{\partial}{\partial\psi},\qquad A=c.$$

$$(\mathbf{10}):\ a_1\mathbf{X_1}+a_2\mathbf{X_2}+a_4\mathbf{X_4}+a_8\mathbf{X_8}+a_9\mathbf{X_9}$$
$$=a_1\frac{\partial}{\partial s}+\left(a_2+a_9\frac{\sqrt{2}rk\cos\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t}+\left(a_9\sin\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r}-\left(a_8\psi\right)\frac{\partial}{\partial z}$$
$$+\left(a_4+a_9\frac{\left(2k^2r^2+1\right)\cos\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi}+\left(a_8z\right)\frac{\partial}{\partial\psi},\qquad A=c,$$

where a_i are arbitrary real constants.

Proof. First, $F_i^\varepsilon : \mathfrak{g}^{\mathrm{I}} \to \mathfrak{g}^{\mathrm{I}}$ defined by $\mathbf{X} \mapsto \mathrm{Ad}(\exp(\varepsilon_i \mathbf{X_i}).\mathbf{X})$ is a linear map, for $i = 1, \dots, 10$. The matrix M_i^ε of F_i^ε, with respect to basis $\{\mathbf{X_1}, \dots, \mathbf{X_{10}}\}$ is

$$M_1^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\varepsilon}{2} & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{\varepsilon}{2} & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \quad M_2^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_3^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\varepsilon & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_4^\varepsilon = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cos\varepsilon & -\sin\varepsilon \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sin\varepsilon & \cos\varepsilon \end{pmatrix},$$

$$M_5^{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_6^{\varepsilon} = \begin{pmatrix} 1 & 0 & -\dfrac{\varepsilon}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\varepsilon & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_7^{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 & 0 & -\dfrac{\varepsilon}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_8^{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\varepsilon & 0 & \sin\varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sin\varepsilon & 0 & \cos\varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cos\varepsilon & \sin\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sin\varepsilon & \cos\varepsilon & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$M_9^{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{\sqrt{2}(\mathcal{V}-2)}{k} & 0 & \dfrac{1}{2}\mathcal{V} & 0 & 0 & 0 & 0 & 0 & \dfrac{\mathcal{W}}{4k} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \dfrac{\sqrt{2}}{2}\mathcal{W} & 0 & k\mathcal{W} & 0 & 0 & 0 & 0 & 0 & \dfrac{1}{2}\mathcal{V} \end{pmatrix},$$

$$M_{10}^{\varepsilon} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{\sqrt{2}(\mathcal{V}-2)}{4k} & 0 & \dfrac{1}{2}\mathcal{V} & 0 & 0 & 0 & 0 & \dfrac{\mathcal{W}}{4k} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \dfrac{\sqrt{2}}{2}\mathcal{W} & 0 & k\mathcal{W} & 0 & 0 & 0 & 0 & \dfrac{1}{2}\mathcal{V} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\mathcal{V} := e^{-2k\varepsilon} + e^{2k\varepsilon}$ and $\mathcal{W} := e^{-2k\varepsilon} - e^{2k\varepsilon}$. In order to classify the one-dimensional Lie subalgebras of Noether symmetries associated to the geodesic Lagrangian (2.18), the following cases are planned such that in each case, by acting a finite number of the adjoint representations M_i^{ε} $(i = 1, \ldots, 10)$ on $\mathbf{X}$, by proper selection of parameters ε_i in each stage, it is gradually tried to make the coefficients of $\mathbf{X}$ vanish and to acquire the most simple form of $\mathbf{X}$.

Let $\mathbf{X} = \sum_{i=1}^{10} a_i \mathbf{X_i}$. Then

$$
\begin{aligned}
&F_{10}^{\varepsilon_{10}} \circ F_9^{\varepsilon_9} \circ \cdots \circ F_1^{\varepsilon_1} \;:\; \mathbf{X} \;\mapsto \\
&\Big[a_1 + \Big(-\frac{1}{2}\varepsilon_6\cos(\varepsilon_8) + \frac{1}{2}\varepsilon_7\sin(\varepsilon_8)\Big)a_3 - \frac{1}{2}\Big(\varepsilon_6\sin(\varepsilon_8) - \varepsilon_7\cos(\varepsilon_8)\Big)a_5\Big]\mathbf{X_1} \\
&+a_2\mathbf{X_2} + \Big[\cos(\varepsilon_8)a_3 + \sin(\varepsilon_8)a_5\Big]\mathbf{X_3} + \Big[a_2\Big(\frac{\sqrt{2}\Big(e^{-2k\varepsilon_9} + e^{2k\varepsilon_9} - 2\Big)}{k} \\
&+\frac{\sqrt{2}\Big(\frac{1}{2}e^{-2k\varepsilon_9} + \frac{1}{2}e^{2k\varepsilon_9}\Big)\Big(e^{-2k\varepsilon_{10}} + e^{2k\varepsilon_{10}} - 2\Big)}{4k}\Big) + a_4\Big(\frac{1}{2}e^{-2k\varepsilon_9} + \frac{1}{2}e^{2k\varepsilon_9}\Big) \\
&\times\Big(\frac{1}{2}e^{-2k\varepsilon_{10}} + \frac{1}{2}e^{2k\varepsilon_{10}}\Big) + \frac{a_9\Big(\frac{1}{2}e^{-2k\varepsilon_9} + \frac{1}{2}e^{2k\varepsilon_9}\Big)\Big(e^{-2k\varepsilon_{10}} - e^{2k\varepsilon_{10}}\Big)}{4k} \\
&+\frac{a_{10}\Big(e^{2k\varepsilon_9} - e^{-2k\varepsilon_9}\Big)}{4k}\Big]\mathbf{X_4} + \Big[-a_3\sin(\varepsilon_8) + a_5\cos(\varepsilon_8)\Big]\mathbf{X_5} \\
&+\Big[\frac{1}{2}\varepsilon_1\cos(\varepsilon_8)a_3 + \frac{1}{2}\varepsilon_1\sin(\varepsilon_8)a_5 + \cos(\varepsilon_8)a_6 + \sin(\varepsilon_8)a_7\Big]\mathbf{X_6} \\
&+\Big[-\frac{1}{2}\varepsilon_1\sin(\varepsilon_8)a_3 + \frac{1}{2}\varepsilon_1\cos(\varepsilon_8)a_5 - \sin(\varepsilon_8)a_6 + \cos(\varepsilon_8)a_7\Big]\mathbf{X_7} \\
&+\Big[a_3\Big(\varepsilon_5\cos(\varepsilon_8) + \varepsilon_3\sin(\varepsilon_8)\Big) + a_5\Big(\varepsilon_5\sin(\varepsilon_8) - \varepsilon_3\cos(\varepsilon_8)\Big) + a_6\Big(\varepsilon_7\cos(\varepsilon_8) \\
&+\varepsilon_6\sin(\varepsilon_8)\Big) + a_7\Big(\varepsilon_7\sin(\varepsilon_8) - \varepsilon_6\cos(\varepsilon_8)\Big) + a_8\Big]\mathbf{X_8} + \Big[a_2\Big(-\frac{\sqrt{2}}{2}\sin(\varepsilon_4) \\
&\times\Big(e^{2k\varepsilon_9} - e^{-2k\varepsilon_9}\Big) - \frac{\sqrt{2}}{4}\sin(\varepsilon_4)\Big(e^{2k\varepsilon_9} - e^{-2k\varepsilon_9}\Big)\Big(e^{2k\varepsilon_{10}} + e^{-2k\varepsilon_{10}} - 2\Big) \\
&-\frac{\sqrt{2}}{2}\cos(\varepsilon_4)\Big(e^{2k\varepsilon_{10}} - e^{-2k\varepsilon_{10}}\Big)\Big) + a_4\Big(-k\sin(\varepsilon_4)\Big(e^{-2k\varepsilon_9} + e^{2k\varepsilon_9}\Big) \\
&\times\Big(\frac{1}{2}e^{-2k\varepsilon_{10}} + \frac{1}{2}e^{2k\varepsilon_{10}}\Big) - k\cos(\varepsilon_4)\Big(e^{2k\varepsilon_{10}} - e^{-2k\varepsilon_{10}}\Big)\Big) \\
&+a_9\Big(-\frac{1}{4}\sin(\varepsilon_4)\Big(-e^{-2k\varepsilon_9} + e^{2k\varepsilon_9}\Big)\Big(e^{-2k\varepsilon_{10}} - e^{2k\varepsilon_{10}}\Big) \\
&+\cos(\varepsilon_4)\Big(\frac{1}{2}e^{2k\varepsilon_{10}} + \frac{1}{2}e^{-2k\varepsilon_{10}}\Big)\Big) - a_{10}\sin(\varepsilon_4)\Big(\frac{1}{2}e^{2k\varepsilon_9} + \frac{1}{2}e^{-2k\varepsilon_9}\Big)\Big]\mathbf{X_9} \\
&+\Big[a_2\Big(-\frac{\sqrt{2}}{2}\cos(\varepsilon_4)\Big(e^{2k\varepsilon_9} - e^{-2k\varepsilon_9}\Big) + \frac{\sqrt{2}}{4}\cos(\varepsilon_4)\Big(e^{2k\varepsilon_9} - e^{-2k\varepsilon_9}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&\times\Big(e^{2k\varepsilon_{10}}+e^{-2k\varepsilon_{10}}-2\Big)-\frac{\sqrt{2}}{2}\sin(\varepsilon_4)\Big(e^{2k\varepsilon_{10}}-e^{-2k\varepsilon_{10}}\Big)\Bigg)\\
&+a_4\Bigg(k\cos(\varepsilon_4)\Big(e^{2k\varepsilon_9}-e^{-2k\varepsilon_9}\Big)\Big(\frac{1}{2}e^{-2k\varepsilon_{10}}+\frac{1}{2}e^{2k\varepsilon_{10}}\Big)\\
&-k\sin(\varepsilon_4)\Big(e^{2k\varepsilon_{10}}-e^{-2k\varepsilon_{10}}\Big)\Bigg)+a_9\Bigg(\frac{1}{4}\cos(\varepsilon_4)\Big(-e^{-2k\varepsilon_9}+e^{2k\varepsilon_9}\Big)\\
&\times\Big(e^{-2k\varepsilon_{10}}-e^{2k\varepsilon_{10}}\Big)+\frac{1}{2}\sin(\varepsilon_4)\Big(e^{2k\varepsilon_{10}}+e^{-2k\varepsilon_{10}}\Big)\Bigg)\\
&+a_{10}\cos(\varepsilon_4)\Big(\frac{1}{2}e^{2k\varepsilon_9}+\frac{1}{2}e^{-2k\varepsilon_9}\Big)\Bigg]\mathbf{X_{10}}
\end{aligned}
$$

Now, we can simplify $\mathbf{X}$ as follows:

If $a_5 \neq 0$ and $a_{10} \neq 0$, then we can make the coefficients of X_3, X_7, X_8, and X_9 vanish by $F_8^{\varepsilon_8}$, $F_1^{\varepsilon_1}$, $F_3^{\varepsilon_3}$, and $F_4^{\varepsilon_4}$. By setting $\varepsilon_8 = -\arctan\left(\frac{a_3}{a_5}\right)$, $\varepsilon_1 = -\frac{2a_7}{a_5}$, $\varepsilon_3 = \frac{a_8}{a_5}$, and $\varepsilon_4 = \arctan\left(\frac{a_9}{a_{10}}\right)$, respectively, and scaling $\mathbf{X}$ if necessary, we can assume that $a_5 = 1$. So, $\mathbf{X}$ is reduced to the case (1).

If $a_5 \neq 0$ and $a_{10} = 0$, then we can make the coefficients of X_3, X_7 and X_8 vanish by $F_8^{\varepsilon_8}$, $F_1^{\varepsilon_1}$, and $F_3^{\varepsilon_3}$. By setting $\varepsilon_8 = -\arctan\left(\frac{a_3}{a_5}\right)$, $\varepsilon_1 = -\frac{2a_7}{a_5}$, and $\varepsilon_3 = \frac{a_8}{a_5}$, respectively, and Scaling $\mathbf{X}$ if necessary, we can assume that $a_5 = 1$. So, $\mathbf{X}$ is reduced to the case (2).

If $a_5 = 0$, $a_3 \neq 0$, and $a_9 \neq 0$, then we can make the coefficients of X_6, X_8 and X_{10} vanish by $F_1^{\varepsilon_1}$, $F_5^{\varepsilon_5}$, and $F_4^{\varepsilon_4}$. By setting $\varepsilon_1 = -\frac{2a_6}{a_3}$, $\varepsilon_5 = -\frac{a_8}{a_3}$, and $\varepsilon_4 = -\arctan\left(\frac{a_{10}}{a_9}\right)$, respectively, and scaling $\mathbf{X}$ if necessary, we can assume that $a_3 = 1$. So, $\mathbf{X}$ is reduced to the case (3).

If $a_5 = 0$, $a_3 \neq 0$ and $a_9 = 0$, then we can make the coefficients of X_6 and X_8 vanish by $F_1^{\varepsilon_1}$ and $F_5^{\varepsilon_5}$. By setting $\varepsilon_1 = -\frac{2a_6}{a_3}$, and $\varepsilon_5 = -\frac{a_8}{a_3}$, respectively, and scaling $\mathbf{X}$ if necessary, we can assume that $a_3 = 1$. So, $\mathbf{X}$ is reduced to the case (4).

If $a_5 = 0$, $a_3 = 0$, $a_7 \neq 0$ and $a_9 \neq 0$, then we can make the coefficients of X_8, X_6, and X_{10} vanish by $F_6^{\varepsilon_6}$, $F_8^{\varepsilon_8}$ and $F_4^{\varepsilon_4}$. By setting $\varepsilon_6 = \frac{a_8}{a_7}$, $\varepsilon_8 = -\arctan\left(\frac{a_6}{a_7}\right)$, and $\varepsilon_4 = -\arctan\left(\frac{a_{10}}{a_9}\right)$, respectively, and scaling $\mathbf{X}$ if necessary, we can assume that $a_7 = 1$. So, $\mathbf{X}$ is reduced to the case (5).

If $a_5 = 0$, $a_3 = 0$, $a_7 \neq 0$, and $a_9 = 0$, then we can make the coefficients of X_8 and X_6 vanish by $F_6^{\varepsilon_6}$ and $F_8^{\varepsilon_8}$. By setting $\varepsilon_6 = \frac{a_8}{a_7}$, and $\varepsilon_8 = -\arctan\left(\frac{a_6}{a_7}\right)$, respectively, and scaling $\mathbf{X}$ if necessary, we can assume that $a_7 = 1$. So, $\mathbf{X}$ is reduced to the case (6).

If $a_5 = 0$, $a_3 = 0$, $a_7 = 0$, $a_6 \neq 0$, and $a_{10} \neq 0$, then we can make the coefficients of X_8 and X_9 vanish by $F_7^{\varepsilon_7}$ and $F_4^{\varepsilon_4}$. By setting $\varepsilon_7 = -\dfrac{a_8}{a_6}$ and $\varepsilon_4 = \arctan\left(\dfrac{a_9}{a_{10}}\right)$, respectively, and scaling $\mathbf{X}$ if necessary, we can assume that $a_6 = 1$. So, $\mathbf{X}$ is reduced to the case (7).

If $a_5 = 0$, $a_3 = 0$, $a_7 = 0$, $a_6 \neq 0$, and $a_{10} = 0$, then we can make the coefficient of X_8 vanish by $F_7^{\varepsilon_7}$. By setting $\varepsilon_7 = -\dfrac{a_8}{a_6}$ and scaling $\mathbf{X}$ if necessary, we can assume that $a_6 = 1$. So, $\mathbf{X}$ is reduced to the case (8).

If $a_5 = 0$, $a_3 = 0$, $a_7 = 0$, $a_6 = 0$, and $a_{10} \neq 0$, then we can make the coefficient of X_9 vanish by $F_4^{\varepsilon_4}$. By setting $\varepsilon_4 = \arctan\left(\dfrac{a_9}{a_{10}}\right)$ and scaling $\mathbf{X}$ if necessary, we can assume that $a_{10} = 1$. So, $\mathbf{X}$ is reduced to the case (9).

If $a_5 = 0$, $a_3 = 0$, $a_7 = 0$, $a_6 = 0$, and $a_{10} = 0$, then $\mathbf{X}$ is reduced to the case (10). □

4. Classification of Killing vector fields

Killing vector fields can be regarded as one of the most significant types of symmetries and are considered as the smooth vector fields, which preserve the metric tensor. These vector fields are extensively applied in various physical fields including in classical mechanics and are closely related to conservation laws. Specifically, remarkable applications of Killing vector fields in relativistic theories are undeniable. The noticeable fact is that the flow corresponding to a Killing vector field generates a symmetry in a way that if each point moves on an object at the same distance in the direction of the Killing vector field then distances on the object will not distorted at all. In particular, a vector field K is a Killing field if the Lie derivative with respect to K of the metric g vanishes. Moreover, the Lie bracket of two Killing vector fields is still a Killing field, and the Killing fields on a manifold M thus form a Lie subalgebra of vector fields on M, which can be considered as the isometry group of the manifold whenever M is complete [11]. Taking into account the significant properties declared above, one naturally expects Killing vectors to be of substantial use in the study of geodesic motion. When one investigates the Lagrangian explaining the motion of a particle, one can realize that Killing vectors are the symmetries of the system and lead to conserved canonical momenta analogous to cyclic coordinates in classical mechanics. Furthermore, one can also try to obtain another conserved quantity related to the spacetime itself if the background metric contains globally well defined Killing vectors.

Let (M, g) be an arbitrary Lorentzian manifold and let $\Im$ be a smooth vector field on M. A curve $\gamma : \mathbb{R} \longrightarrow M$ whose tangent vector at every point $p \in \gamma$ is equal to $\Im$, is denoted by an integral curve of $\Im$. In other words, for all smooth functions $f : M \longrightarrow \mathbb{R}$, the following relation is satisfied: $\Im_p(f) = \dfrac{d}{d\wp}\Big(f o \gamma(\wp)\Big)\Big|_p$, where $\wp$ parameterizes the curve γ. For a given local coordinate system x^μ on M, this is equivalent to that the components of $\Im$ in that coordinate system must satisfy $\Im^{\mu}_{p} = \frac{d}{d\wp} x^{\mu}\big(\gamma(\wp)\big)\Big|_{p}$. Taking into account the fact that every point $p \in M$ lies on a unique integral curve, the set of integral curves create a congruence whenever $\Im$ is smooth and everywhere nonzero. Significantly, for a given congruence a one-parameter family of diffeomorphisms from M onto itself can be associated, which is described as follows: Corresponding to each $s \in \mathbb{R}$, designate a map $\mathcal{F}_s : M \longrightarrow M$, where $\mathcal{F}_s(p)$ is the point parameter distance s from p along $\Im$; that is, if $p = \gamma(\wp_0)$, then $\mathcal{F}_s(p) = \gamma(\wp_0 + s)$. Furthermore, from the algebraic point of view, considering the composition law $\mathcal{F}_s o \mathcal{F}_t = \mathcal{F}_{s+t}$, the identity $\mathcal{F}_0$, and the inverse $(\mathcal{F}_s)^{-1} = \mathcal{F}_{-s}$, these transformations construct an abelian group. Thus the notion of the Lie derivative $\mathcal{L}_{\Im}$ along the vector field $\Im$ is created. When applied at a point p to a vector K it is defined by the following identity:

$$\big(\mathcal{L}_{\Im} K\big)_p = \lim_{\delta\wp \longrightarrow 0} \frac{K_p - (\mathcal{F}_{\delta\wp})_* K_{\mathcal{F}_{-\delta\wp}}(p)}{\delta\wp} \tag{4.1}$$

where $(\mathcal{F}_s)_*$ projects a vector defined at p to a vector defined at $\mathcal{F}_s(p)$ and is denoted by the push-forward corresponding to the group element $\mathcal{F}_s$. Moreover, it can be illustrated that the Lie derivative of a vector is equal to the bracket: $\big(\mathcal{L}_{\Im} K\big)_p = [\Im, K]_p$, where $[X, Y]^{\mu} = X^{\nu} Y^{\mu}_{,\nu} - Y^{\nu} X^{\mu}_{,\nu}$. Analogous to (4.1), the Lie derivative can be applied appropriately to any tensor on M. Specifically to a metric tensor g on M the Lie derivative is defined by

$$\big(\mathcal{L}_{\Im} g\big)_p = \lim_{\delta\wp \longrightarrow 0} \frac{g_p - (\mathcal{F}_{\delta\wp})^* g_{\mathcal{F}_{\delta\wp}}(p)}{\delta\wp}. \tag{4.2}$$

The remarkable fact is that the Lie derivative of g entails the pull-back $\mathcal{F}^*_s$, which maps a covector at $\mathcal{F}_s(p)$ to a covector at p, mainly due to the fact that the components of g transform covariantly. It can be demonstrated that

$$\big(\mathcal{L}_{\Im} g\big)_{\mu\nu} = \nabla_{\mu} \Im_{\nu} + \nabla_{\nu} \Im_{\mu}. \tag{4.3}$$

Meanwhile, if the metric does not change under the transformation $\mathcal{F}_s$, then the transformation is called an isometry and the metric possesses a symmetry. In this case, $\mathcal{L}_{\Im} g = 0$, which leads to the following identity:

$$\nabla_{\mu} \Im_{\nu} + \nabla_{\nu} \Im_{\mu} = 0. \tag{4.4}$$

This relation is denoted by Killing's equation and a vector $\Im$ that satisfies (4.4) is called a Killing vector. It is noticeable that this identity contains the metric implicity, which is hidden in ∇. In addition, the symmetries of a spacetime explicitly leads to determining the vectors, which satisfy the Killing equation; this can be thoroughly fulfilled either by inspection or via integrating (4.4). An isometry is a distance preserving mapping among different spaces. In the case of a Lorentzian manifold (M, g), the transformed metric $\tilde{g}^{ij}(\tilde{x})$ has to be the similar function of its argument $\tilde{x}$ as the original metric $g^{ij}(x)$ of its argument x, that is, $\tilde{g}^{ij}(x) = g^{ij}(x)$. The metric under an arbitrary transformation transforms as follows:

$$\tilde{g}^{ij}(\tilde{x}) = \frac{\partial \tilde{x}^i}{\partial x^k} \frac{\partial \tilde{x}^j}{\partial x^l} g^{kl}(x) \quad \text{or} \quad g^{ij}(x) = \frac{\partial x^i}{\partial \tilde{x}^k} \frac{\partial x^j}{\partial \tilde{x}^l} \tilde{g}^{kl}(\tilde{x}). \tag{4.5}$$

For an isometry, the following equality is satisfied:

$$\tilde{g}^{ij}(x) = g^{ij}(x) = \frac{\partial x^i}{\partial \tilde{x}^k}\frac{\partial x^j}{\partial \tilde{x}^l}\tilde{g}^{kl}(\tilde{x}). \tag{4.6}$$

Now, by imposing an infinitesimal coordinate transformation $\tilde{x}^i = x^i + \varepsilon\Im(x^k)$, $\varepsilon << 1$ on the metric, we have

$$\begin{aligned}\tilde{g}^{ij}(\tilde{x}) &= \frac{\partial \tilde{x}^i}{\partial x^k}\frac{\partial \tilde{x}^j}{\partial x^l}g^{kl}(x) = \left(\delta^i_k + \varepsilon\Im^i_{,k}\right)\left(\delta^l_j + \varepsilon\Im^j_{,l}\right)g^{kl}(x)\\ &= g^{ij}(x) + \varepsilon\left(\Im^{i,j} + \Im^{j,i}\right) + O(\varepsilon^2).\end{aligned} \tag{4.7}$$

A Taylor expansion results

$$\tilde{g}^{ij}(\tilde{x}) = \tilde{g}^{ij}(x + \varepsilon\Im) = \tilde{g}^{ij}(x) + \varepsilon\Im^k\partial_k\tilde{g}^{ij}(x) + O(\varepsilon^2). \tag{4.8}$$

Setting equal relations (4.7) and (4.8), it is deduced that

$$g^{ij}(x) + \varepsilon\left(\Im^{i,j} + \Im^{j,i}\right) = \tilde{g}^{ij}(x) + \varepsilon\Im^k\partial_k\tilde{g}^{ij}(x). \tag{4.9}$$

Moreover, it is observed that the metric is kept invariant if

$$\Im^{i,j} + \Im^{j,i} - \Im^k\partial_k g^{ij} = 0, \tag{4.10}$$

or equivalently,

$$g^{jk}\partial_k\Im^i + g^{ik}\partial_k\Im^j - \Im^k\partial_k g^{ij} = 0. \tag{4.11}$$

Consequently, the terms involving connection coefficients totally vanish via expressing partial derivatives as covariant ones. Subsequently, taking into account the identity (4.11), the Killing equations (4.4) are deduced again. The vectors $\Im$ satisfying (4.4) are called Killing vectors of the metric. Hence, moving along a Killing vector field, the metric is preserved invariant. Although equation (4.11) involves only partial derivatives, it is also invariant under arbitrary coordinate transformations mainly due to the fact that relation (4.4) is tensor equation. As mentioned above, without the connection requirement, one can define the derivative of an arbitrary tensor along a vector field denoted by Lie derivative. In classical physics, the presence of symmetries is closely related to the existence of conservation laws. In the following, we analyze the geodesic motion of test particles [6, 33]. Assume the action of a particle in a spacetime (M, g), which is moving on a curve γ with parameter $\wp$ and endpoints A and B. Select a coordinate system x^μ and designate the coordinates of the curve by $x^\mu(\wp)$. Then the action for γ is defined by

$$I(x^\mu) = m\int d\tau = m\int_{\wp_A}^{\wp_B}\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\wp}\frac{dx^\nu}{d\wp}}. \tag{4.12}$$

If the curve is deformed by a small amount $\delta x^\mu(\wp)$ and the action is required to be stationary with respect to the declared variation, then the Euler–Lagrange equations are resulted as follows:

$$\frac{\delta I}{\delta x^\mu} = 0 \Longrightarrow \nabla_{(\wp)}\dot{x}^\mu \equiv \frac{d}{d\wp}\dot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma \ \propto \ \dot{x}^\mu. \tag{4.13}$$

The dot indicates differentiation with respect to $\wp$. Furthermore, the solutions of (4.13) are the geodesics of (M, g). In particular, if the right hand side of above equation vanishes, that is, $\nabla_{(\wp)}\dot{x}^\mu = 0$, then the geodesics is called to be affinely parameterized and $\wp$ is considered as an affine parameter. Besides, if we insert $\wp = \tau$, then (4.13) simplifies to

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0. \tag{4.14}$$

In order to thoroughly determine the geodesic curves, we can cast (4.12) into a more effective structure via inserting an independent function $e(\wp)$ denoted by the auxiliary field:

$$I(x^\mu, e) = \frac{1}{2}\int_{\wp_A}^{\wp_B} d\wp \,\Big[e^{-1}(\wp)g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2 e(\wp)\Big]. \tag{4.15}$$

It is noticeable that from $\dfrac{\delta I}{\delta e} = 0$ we obtain

$$-e^{-2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - m^2 = 0 \Longrightarrow e = \frac{1}{m}\sqrt{-g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} = \frac{1}{m}\frac{d\tau}{d\wp} \tag{4.16}$$

and from $\dfrac{\delta I}{\delta x^\mu}$ the following identity is deduced:

$$\frac{d}{d\wp}\dot{x}^\mu + \Gamma^\mu_{\rho\sigma}\dot{x}^\rho\dot{x}^\sigma = \big(e^{-1}\dot{e}\big)\dot{x}^\mu. \tag{4.17}$$

Thus the equivalence of (4.12) to (4.15) is illustrated. Moreover, it is explicitly observed that relation (4.17) is precisely the geodesic equation $\nabla_{(\wp)}\dot{x}^\mu = (e^{-1}\dot{e})\dot{x}^\mu$ and (4.16) relates the auxiliary field e to the choice of parameter $\wp$. Suppose now an infinitesimal translation of the curve γ along a Killing vector field K, that is, leaving the auxiliary field unchanged. In the coordinate chart x^μ, this associates with $x^\mu \longrightarrow x^\mu + \alpha K^\mu$, where α is an infinitesimal constant. Accordingly, due to (4.15), the action will be varied as follows:

$$\begin{aligned}\delta I &= I\big(x^\mu + \alpha K^\mu, e\big) - I\big(x^\mu, e\big)\\ &= \frac{\alpha}{2}\int d\wp\Big[e^{-1}\Big(g_{\mu\nu}\dot{K}^\mu\dot{x}^\nu + g_{\mu\nu}\dot{x}^\mu\dot{K}^\nu + g_{\mu\nu,\sigma}\dot{x}^\mu\dot{x}^\nu K^\sigma\Big)\Big].\end{aligned} \tag{4.18}$$

Taking into account the fact that $\dot{K}^\mu = \dfrac{d}{d\tau}K^\mu = \dfrac{dx^\nu}{d\tau}\partial_\nu K^\mu = \dot{x}^\nu K^\mu_{,\nu}$, we have

$$\begin{aligned}\delta I &= \frac{\alpha}{2}\int d\wp\Big[e^{-1}\Big(g_{\mu\nu}\dot{x}^\sigma\dot{x}^\nu K^\mu_{,\sigma} + g_{\mu\nu}\dot{x}^\mu\dot{x}^\sigma K^\nu_{,\sigma} + g_{\mu\nu,\sigma}\dot{x}^\mu\dot{x}^\nu K^\sigma\Big)\Big]\\ &= \frac{\alpha}{2}\int d\wp\Big[e^{-1}\dot{x}^\mu\dot{x}^\nu\Big(g_{\sigma\nu}K^\sigma_{,\mu} + g_{\mu\sigma}K^\sigma_{,\nu} + g_{\mu\nu,\sigma}K^\sigma\Big)\Big].\end{aligned} \tag{4.19}$$

Since the vector field K satisfies the Killing's equation, we have:

$$\delta I = \frac{\alpha}{2}\int d\wp\Big[e^{-1}\dot{x}^\mu\dot{x}^\nu\Big(\nabla_\mu K_\nu + \nabla_\nu K_\mu\Big)\Big] = 0. \tag{4.20}$$

Consequently, it is demonstrated that K being Killing results a symmetry of the particle action. Significantly, it can be proved that corresponding to this symmetry there exists a quantity (charge), which is totally conserved along the geodesic curves. Assume that K is a Killing vector field and that $\delta x^\mu = \alpha K^\mu$ is a small variation generated by K. As discussed above, such variations leave the action invariant, that is, $\delta I = 0$. For simplicity, we set $\wp = \tau$, and introducing the Lagrangian density L as $I = \displaystyle\int L d\wp$, then we have

$$\frac{\partial L}{\partial x^\mu}\delta x^\mu + \frac{\partial L}{\partial \dot{x}^\mu}\delta \dot{x}^\mu = 0. \tag{4.21}$$

Now, by applying the Euler–Lagrange equations, $\dfrac{\partial L}{\partial x^\mu} = \dfrac{d}{d\tau}\dfrac{\partial L}{\partial \dot{x}^\mu} = \dot{p}_\mu$, where p_μ is the momentum of the particle and according to (4.16) is defined by $p_\mu = \dfrac{\partial L}{\partial \dot{x}^\mu} = e^{-1} g_{\mu\nu}\dot{x}^\nu = m g_{\mu\nu}\dfrac{dx^\nu}{d\tau}$. Therefore, the equation (4.21) can be rewritten as follows:

$$0 = \dot{p}_\mu \alpha K^\mu + p_\mu \alpha \dot{K}^\mu = \alpha \frac{d}{d\tau}\big(K^\mu p_\mu\big) = \alpha \frac{d}{d\tau}\mathcal{Q}. \tag{4.22}$$

As a consequence, it is proved that the quantity $\mathcal{Q} = k^\mu p_\mu$ is conserved along geodesics. This fact can be considered as a particular case of a much more general theorem of Noether, which states that symmetries in a typical variational principle lead to conservation laws (refer to [6, 11, 33] for more details). In the following, an exhaustive analysis of Killing vector fields for the metric (1.21) is presented by re-expressing the metric in the orthogonal frame.

4.1. **Computation of the Killing vector fields for solution (1.21).** In this section, we apply an orthogonal frame to obtain the Killing vector fields for the metric (1.21). First of all, we set up a five-dimensional spacetime with coordinates $\big[t, r, z, \phi, \psi\big]$ denoted by ϖ^{I} given by

$$\varpi^{\mathrm{I}} = \left[\frac{1}{\sqrt{k^2r^2+1}}dr,\ dz,\ r\sqrt{k^2r^2+1}dx,\ dy,\ dt + \sqrt{2}kr^2 dx\right]. \tag{4.23}$$

Then we define a coframe and calculate the structure equations for this coframe as follows:

$$\begin{aligned} &d\Theta_1 = 0, \quad d\Theta_2 = 0, \qquad d\Theta_3 = \frac{2k^2r^2+1}{r\sqrt{k^2r^2+1}}\,\Theta_1 \wedge \Theta_3, \\ &d\Theta_4 = 0, \qquad d\Theta_5 = 2\sqrt{2}k\ \Theta_1 \wedge \Theta_3. \end{aligned} \tag{4.24}$$

Taking into account the pentad (1.10), the metric (1.21) is expressed by (1.11) in the orthogonal frame. Subsequently, the following seven Killing vectors for the metric (1.21) are resulted in the adapted frame:

$$
\begin{cases}
(\mathbf{1}) : \mathbf{K_1} = -\dfrac{\sqrt{2}\cos(x)}{4k}\ \mathbf{E_1} + \dfrac{\sqrt{2}\sin(x)\Big(2k^2r^2+1\Big)}{4k}\ \mathbf{E_3} + r\mathcal{Q}\sin(x)\ \mathbf{E_5}, \\
(\mathbf{2}) : \mathbf{K_2} = \dfrac{\sqrt{2}\sin(x)}{4k}\ \mathbf{E_1} + \dfrac{\sqrt{2}\cos(x)\Big(2k^2r^2+1\Big)}{4k}\ \mathbf{E_3} + r\mathcal{Q}\cos(x)\ \mathbf{E_5}, \\
(\mathbf{3}) : \mathbf{K_3} = \mathbf{E_5}, \\
(\mathbf{4}) : \mathbf{K_4} = \dfrac{r\sqrt{2k^2r^2+2}}{2k}\ \mathbf{E_3} + r^2\ \mathbf{E_5}, \\
(\mathbf{5}) : \mathbf{K_5} = y\ \mathbf{E_2} - z\ \mathbf{E_4}, \\
(\mathbf{6}) : \mathbf{K_6} = -\mathbf{E_4}, \\
(\mathbf{7}) : \mathbf{K_7} = -\mathbf{E_2},
\end{cases}
\tag{4.25}
$$

where $\mathcal{Q} := \sqrt{k^2r^2+1}$. Furthermore, here are the structure equations for the Lie algebra of Killing vectors denoted by $\mathcal{K}^{\mathrm{I}}$:

$$
\begin{aligned}
&[\mathbf{K_1}, \mathbf{K_2}] = -\frac{\sqrt{2}}{2}k\ \mathbf{K_4} - \frac{\sqrt{2}}{4k}\ \mathbf{K_3}, \qquad [\mathbf{K_1}, \mathbf{K_4}] = -\frac{\sqrt{2}}{2k}\ \mathbf{K_2}, \\
&[\mathbf{K_2}, \mathbf{K_4}] = \frac{\sqrt{2}}{2k}\ \mathbf{K_1}, \qquad [\mathbf{K_5}, \mathbf{K_6}] = -\mathbf{K_7}, \qquad [\mathbf{K_5}, \mathbf{K_7}] = \mathbf{K_6}.
\end{aligned}
\tag{4.26}
$$

Significantly, by considering the following basis for the original Lie algebra of Killing vector fields $\mathcal{K}^{\mathrm{I}}$, it will decompose into an internal direct sum of subalgebras, where each summand is indecomposable,

$$
\Big\{\mathbf{F_1}, \mathbf{F_2}, \mathbf{F_3}, \mathbf{F_4}, \mathbf{F_5}, \mathbf{F_6}, \mathbf{F_7}\Big\} := \Big\{\mathbf{K_1}, \mathbf{K_2}, \mathbf{K_3} + 2k^2\ \mathbf{K_4}, \mathbf{K_5}, \mathbf{K_6}, \mathbf{K_7}, \mathbf{K_3}\Big\}.
$$

The expression of $\mathcal{K}^{\mathrm{I}}$ in this new basis described above, will be denoted by $\tilde{\mathcal{K}}^{\mathrm{I}}$. Meanwhile, $\mathcal{A}^{\mathrm{I}}$ is a matrix, which defines a Lie algebra isomorphism from $\mathcal{K}^{\mathrm{I}}$ to $\tilde{\mathcal{K}}^{\mathrm{I}}$ (the Lie algebra defined by the direct sum of indecomposable Lie subalgebras) given by

$$
\mathcal{A}^{\mathrm{I}} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2k^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & -\frac{1}{2k^2} & 0 & 0 & 0
\end{pmatrix}
$$

The commutator table of $\tilde{\mathcal{K}}^{\mathrm{I}}$ is illustrated in Table 3, where the entry in the ith row and jth column is defined as $[F_i, F_j] = F_iF_j - F_jF_i, \quad i, j = 1, \ldots, 7$.

TABLE 3. Commutation relations satisfied by infinitesimal generators for the Lie algebra $\tilde{\mathcal{K}}^{\mathrm{I}}$

	$\mathbf{F_1}$	$\mathbf{F_2}$	$\mathbf{F_3}$	$\mathbf{F_4}$	$\mathbf{F_5}$	$\mathbf{F_6}$	$\mathbf{F_7}$
$\mathbf{F_1}$	0	$-\frac{\sqrt{2}}{4k}\ \mathbf{F_3}$	$-\sqrt{2}k\ \mathbf{F_2}$	0	0	0	0
$\mathbf{F_2}$	$-\frac{\sqrt{2}}{4k}\ \mathbf{F_3}$	0	$\sqrt{2}k\ \mathbf{F_1}$	0	0	0	0
$\mathbf{F_3}$	$\sqrt{2}k\ \mathbf{F_2}$	$-\sqrt{2}k\ \mathbf{F_1}$	0	0	0	0	0
$\mathbf{F_4}$	0	0	0	0	$-\mathbf{F_6}$	$\mathbf{F_5}$	0
$\mathbf{F_5}$	0	0	0	$\mathbf{F_6}$	0	0	0
$\mathbf{F_6}$	0	0	0	$-\mathbf{F_5}$	0	0	0
$\mathbf{F_7}$	0	0	0	0	0	0	0

5. Computation of the conservation laws via Noether's theorem

A significant systematic way of determining conservation laws for systems of Euler–Lagrange equations whenever their Noether symmetries are known is via Noether's theorem [25]. This theorem is fundamentally relied on the availability of a Lagrangian and the corresponding Noether symmetries, which leave the action integral invariant.

Consider the action integral $J[U]$ (2.3) and an infinitesimal change of U given by $U(x) \longrightarrow U(x) + \varepsilon v(x)$, where $v(x)$ is an arbitrary function such that $v(x)$ and its derivatives to order $k-1$ vanish on the boundary $\partial\Omega$ of the domain Ω. Thus the corresponding variation in the Lagrangian $L[U]$ is expressed as follows [3]:

$$\begin{aligned} \delta L &= L\Big(x, U + \varepsilon v, \partial U + \varepsilon \partial v, \ldots, \partial^k U + \varepsilon \partial^k v\Big) - L\Big(x, U, \partial U, \ldots, \partial^k U\Big) \\ &= \varepsilon \left(\frac{\partial L\big[U\big]}{\partial U^\sigma}\ v^\sigma + \frac{\partial L\big[U\big]}{\partial U^\sigma_j}\ v^\sigma_j + \cdots + \frac{\partial L\big[U\big]}{\partial U^\sigma_{j_1\cdots j_k}}\ v^\sigma_{j_1\cdots j_k} \right) + O\big(\varepsilon^2\big). \end{aligned} \tag{5.1}$$

Then after applying integration by parts repeatedly, it can be illustrated that

$$\delta L = \varepsilon \Big(v^\sigma \mathrm{E}_{U^\sigma}\big(L\big[U\big]\big) + D_i W^i\big[U, v\big] \Big) + O\big(\varepsilon^2\big), \tag{5.2}$$

where E_{U^σ} is the Euler operator with respect to U^σ given by

$$E_{U^\sigma} = \frac{\partial}{\partial U^\sigma} - D_j \frac{\partial}{\partial U^\sigma_j} + \cdots + (-1)^k D_{j_1} \cdots D_{j_k} \frac{\partial}{\partial U^\sigma_{j_1\cdots j_k}} + \cdots \tag{5.3}$$

and

$$W^i\big[U,v\big]=v^\sigma\left(\frac{\partial L\big[U\big]}{\partial U_i^\sigma}+\cdots+(-1)^{k-1}D_{j_1}\cdots D_{j_{k-1}}\frac{\partial L\big[U\big]}{\partial U^\sigma_{ij_1\cdots j_{k-1}}}\right)$$
$$+v^\sigma_{j_1}\left(\frac{\partial L\big[U\big]}{\partial U_i^\sigma}+\cdots+(-1)^{k-2}D_{j_2}\cdots D_{j_{k-1}}\frac{\partial L\big[U\big]}{\partial U^\sigma_{ij_1j_2\cdots j_{k-1}}}\right) \tag{5.4}$$
$$+\cdots+v^\sigma_{j_1\cdots j_{k-1}}\frac{\partial L\big[U\big]}{\partial U^\sigma_{ij_1j_2\cdots j_{k-1}}}.$$

According to identity (5.2) and the divergence theorem, the corresponding variation in the action integral $J[U]$ (2.3) is defined as follows [3]:

$$\begin{aligned}\delta J&=J\big[U+\varepsilon v\big]-J\big[U\big]=\int_\Delta \delta L dx\\&=\varepsilon\int_\Delta\Big(v^\sigma \mathrm{E}_{U^\sigma}\big(L\big[U\big]\big)+D_lW^l\big[U,v\big]\Big)+O\big(\varepsilon^2\big)\\&=\varepsilon\left(\int_\Delta\Big(v^\sigma \mathrm{E}_{U^\sigma}\big(L\big[U\big]\big)dx+\int_{\partial\Delta}W^l\big[U,v\big]n^l dS\right)+O\big(\varepsilon^2\big),\end{aligned} \tag{5.5}$$

where $n=\big(n^1,\ldots,n^n\big)$ is the unit outward normal vector to the boundary $\partial\Delta$ and $\displaystyle\int_{\partial\Delta}$ indicates the surface integral over $\partial\Delta$. Therefore, it follows that if $U=U(x)$ extremizes the action integral (2.3), then for an arbitrary $v(x)$ defined on the domain Δ, the $O(\varepsilon)$ term of ∂J must vanish, and as a consequence, we have

$$\int_\Delta v^\sigma \mathrm{E}_{u^\sigma}\Big(L\big[u\big]\Big)dx=0. \tag{5.6}$$

Ultimately, it is deduced that if $U=u(x)$ extremizes the action integral $J[U]$ (2.3), then $u(x)$ must satisfy the following partial differential equation system:

$$\mathrm{E}_{u^\sigma}\Big(L\big[U\big]\Big)=\frac{\partial L\big[U\big]}{\partial u^\sigma}+\cdots+(-1)^kD_{j_1}\cdots D_{j_k}\frac{\partial L\big[U\big]}{\partial U^\sigma_{j_1\cdots j_k}}=0,\ \sigma=1,\ldots,m. \tag{5.7}$$

Equations (5.7) are denoted by the Euler–Lagrange equations satisfied by an extremum $U=u(x)$ of the action integral (2.3). The one-parameter Lie group of point transformations (2.1) is equivalent to the one-parameter family of local transformations:

$$\begin{cases}\big(x^*\big)^i=x^i, \qquad i=1,\ldots,n,\\ \big(U^*\big)^\mu=U^\mu+\varepsilon\big[\Omega^\mu(x,U)-U_i^\mu\xi^i(x,U)\big]+O(\varepsilon^2), \qquad \mu=1,\ldots,m,\end{cases} \tag{5.8}$$

The infinitesimal generator for the one-parameter family of transformations (5.8) is defined by

$$\hat{X}=\Big[\Omega^\mu(x,U)-U_i^\mu\xi^i(x,U)\Big]\frac{\partial}{\partial U^\mu}. \tag{5.9}$$

and $\hat{X}^{(k)}$ denotes the corresponding kth prolonged infinitesimal generator. By imposing the transformation (5.8), the corresponding infinitesimal variation $U(x) \longrightarrow U(x) + \varepsilon v(x)$ has components, $v^\mu(x) = \hat{\Omega}^\mu[U] = \Omega^\mu(x, U) - U_i^\mu \xi^i(x, U)$ in terms of the transformation (5.8). Furthermore, from the group property of (5.8) it is deduced that

$$\delta L = \varepsilon \hat{X}^{(k)} L[U] + O(\varepsilon^2) \Longrightarrow \int_\Delta \delta L dx = \varepsilon \int_\Delta \hat{X}^{(k)} L[U] dx + O(\varepsilon^2). \quad (5.10)$$

Accordingly, after comparing expression (5.10) to identity (5.5) with $v^\mu(x) = \hat{\Omega}^\mu[U] = \Omega^\mu(x, U) - U_i^\mu \xi^i(x, U)$, the following significant identity is resulted [3]:

$$\hat{X}^{(k)} L[U] \equiv \hat{\Omega}^\mu[U] E_{U^\mu}(L[U]) + D_i W^i\big[U, \hat{\Omega}[U]\big], \quad (5.11)$$

where $W^i\big[U, \hat{\Omega}[U]\big]$ is expressed by (5.4) with the trivial substitutions. Now, assume that $X^{(k)}$ is the kth prolonged infinitesimal generator of the one-parameter Lie group of point transformations (2.1) and that $\hat{X}^{(k)}$ is the kth prolonged infinitesimal generator of the equivalent one-parameter family of transformations (5.8). Then for an arbitrary function $F[U] = F\big(x, U, \partial U, \ldots, \partial^k U\big)$, the following relation is satisfied:

$$X^{(k)} F[U] + F[U] D_i \xi^i(x, U) \equiv \hat{X}^{(k)} F[U] + D_i\big(F[U]\xi^i(x, U)\big). \quad (5.12)$$

Now, by inserting $F[U] = L[U]$ in the identity (5.12) and from (2.8), for arbitrary functions $U(x)$, we have

$$\hat{X}^{(k)} L[U] + D_i\big(L[U]\xi^i(x, U)\big) \equiv 0. \quad (5.13)$$

Thus substitution for $\hat{X}^{(k)} L[U]$ in (5.13) through (5.11) yields the following identity:

$$\hat{\Omega}^\mu[U] E_{U^\mu}\big(L[U]\big) \equiv -D_i\big(\xi^i(x, U)\big) L[U] + W^i\big[U, \hat{\Omega}[U]\big]. \quad (5.14)$$

In other words, $\big\{\hat{\Omega}^\mu[U]\big\}_{\mu=1}^m$ is a set of local multipliers of the Euler–Lagrange system (5.6). Then the left hand side of the identity (5.14) vanishes. Ultimately, this yields the following local conservation law

$$D_i\bigg(\xi^i(x, U) L[u] + W^i\big[u, \hat{\Omega}[u]\big]\bigg) = 0 \quad (5.15)$$

for any solution $u = \Theta(x)$ of the Euler–Lagrange system (5.6) (refer to [3] for more complete details). Specifically, if $\mathbf{X}$ (2.12) is a Noether point symmetry corresponding to the Lagrangian $L(s, x^\mu, \dot{x}^\mu)$, then

$$T = \xi L + (\Omega^\mu - \dot{x}^\mu \xi) \frac{\partial L}{\partial \dot{x}^\mu} - A \quad (5.16)$$

is a first integral of (2.9) corresponding to $\mathbf{X}$, where $A = A(s, x^\mu)$ is the gauge function [13, 15].

Now, by applying identity (5.16), we will compute all the conserved flows corresponding to the resulted Noether symmetries for solution (1.21). Moreover, in the following, the conserved flows associated to those infinitesimal generators obtained via constructing an optimal system of one-dimensional subalgebras of the Lie algebra of Noether symmetries (as demonstrated in Theorem 3.1) are calculated. Each of these resulted conserved quantities yields a conservation law for the system of geodesic equations.

5.1. **Computation of the Noether conservation laws for solution (1.21).** In this section, first of all, we will compute all the conserved flows corresponding to the Noether symmetries $\mathbf{X_1}, \ldots, \mathbf{X_{10}}$ resulted in Corollary 2.3. Each of these conserved quantities yields a conservation law for the system of geodesic equations (2.19). For example, for the Noether symmetry $\mathbf{X_1} = \dfrac{\partial}{\partial s}$, we get the following conserved vector:

$$\begin{aligned} T^1 &= \xi L + (\Omega^1 - \dot{t}\xi)\tfrac{\partial L}{\partial \dot{t}} + (\Omega^2 - \dot{r}\xi)\tfrac{\partial L}{\partial \dot{r}} + (\Omega^3 - \dot{z}\xi)\tfrac{\partial L}{\partial \dot{z}} \\ &\quad + (\Omega^4 - \dot{\varphi}\xi)\tfrac{\partial L}{\partial \dot{\phi}} + (\Omega^5 - \dot{\psi}\xi)\tfrac{\partial L}{\partial \dot{\psi}} - A \\ &= -\dot{t}^2 + \frac{\dot{r}^2}{1+k^2r^2} + \dot{z}^2 - (k^2r^4 - r^2)\dot{\phi}^2 - 2\sqrt{2}kr^2\dot{t}\dot{\phi} + \dot{\psi}^2 - c. \end{aligned}$$

Similarly, we have computed the conserved vectors corresponding to the other Noether symmetries. The results are presented in Table 4. In the following,

Table 4. Conservation laws of (2.19) resulted from the Noether's theorem

	Noether Symmetry	Conserved Vectors
1	$\mathbf{X_1} = \partial s$	$T^1 = -\dot{t}^2 + \dfrac{\dot{r}^2}{1+k^2r^2} + \dot{z}^2 - (k^2r^4 - r^2)\dot{\phi}^2 - 2\sqrt{2}kr^2\dot{t}\dot{\phi} + \dot{\psi}^2 - c$
2	$\mathbf{X_2} = \partial t$	$T^2 = 2\dot{t} + 2\sqrt{2}\,kr^2\,\dot{\phi} - c$
3	$\mathbf{X_3} = \partial z$	$T^3 = -2\dot{z} - c$
4	$\mathbf{X_4} = \partial \phi$	$T^4 = 2(k^2r^4 - r^2)\dot{\phi} + 2\sqrt{2}\,kr^2\dot{t} - c$
5	$\mathbf{X_5} = \partial \psi$	$T^5 = -2\dot{\psi} - c$
6	$\mathbf{X_6} = -\frac{1}{2}s\partial z$	$T^6 = s\dot{z} - z - c$
7	$\mathbf{X_7} = -\frac{1}{2}s\partial \psi$	$T^7 = s\dot{\psi} - \psi - c$
8	$\mathbf{X_8} = -\psi\partial z + z\partial\psi$	$T^8 = 2\psi\dot{z} - 2z\dot{\psi} - c$
9	$\mathbf{X_9} = \dfrac{\sqrt{2}\,kr\cos\phi}{\sqrt{1+k^2r^2}}\partial t + \sqrt{1+k^2r^2}\,\sin\phi\,\partial r + \dfrac{(2k^2r^2+1)\cos\phi}{r\sqrt{1+k^2r^2}}\,\partial\phi$	$T^9 = \dfrac{\sqrt{2}\,rk\cos\phi(2\dot{t} + 2\sqrt{2}\,kr^2\dot{\phi})}{\sqrt{1+k^2r^2}} - \dfrac{2\sin\phi\sqrt{1+k^2r^2}}{1+k^2r^2}\,\dot{r} + \dfrac{2(2k^2r^2+1)\cos\phi\big((k^2r^4 - r^2)\dot{\phi} + \sqrt{2}\,kr^2\dot{t}\big)}{r\sqrt{1+k^2r^2}} - c$
10	$\mathbf{X_{10}} = -\dfrac{\sqrt{2}\,kr\sin\phi}{\sqrt{1+k^2r^2}}\partial t + \sqrt{1+k^2r^2}\,\cos\phi\,\partial r - \dfrac{(2k^2r^2+1)\sin\phi}{r\sqrt{1+k^2r^2}}\,\partial\phi$	$T^{10} = -\dfrac{\sqrt{2}\,rk\sin\phi(2\dot{t} + 2\sqrt{2}\,kr^2\dot{\phi})}{\sqrt{1+k^2r^2}} - \dfrac{2\cos\phi\sqrt{1+k^2r^2}}{1+k^2r^2}\,\dot{r} - \dfrac{2(2k^2r^2+1)\sin\phi\big((k^2r^4 - r^2)\dot{\phi} + \sqrt{2}\,kr^2\dot{t}\big)}{r\sqrt{1+k^2r^2}} - c$

the conserved flows associated to those infinitesimal generators obtained via constructing an optimal system of one-dimensional subalgebras of the Lie algebra of Noether symmetries (as demonstrated in Theorem 3.1) are calculated.

(**1**) : For the symmetry operator,

$$a_1\mathbf{X_1}+a_2\mathbf{X_2}+a_4\mathbf{X_4}+\mathbf{X_5}+a_6\mathbf{X_6}+a_{10}\mathbf{X_{10}}$$
$$=a_1\frac{\partial}{\partial s}+\left(a_2-a_{10}\frac{\sqrt{2}rk\sin\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t}+\left(a_{10}\cos\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r}-a_6\big(\frac{s}{2}\big)\frac{\partial}{\partial z}$$
$$+\left(a_4-a_{10}\frac{\left(2k^2r^2+1\right)\sin\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi}+\frac{\partial}{\partial\psi},\quad A=z+c.$$

By applying Noether's theorem, the following conservation law is resulted:

$$D_s\Bigg(a_1\left(\dot{t}^2-\frac{\dot{r}^2}{\mathcal{Q}^2}-\dot{z}^2+\left(2k^2r^4-r^2\mathcal{Q}^2\right)\right)\dot{\varphi}^2+2\sqrt{2}kr^2\dot{t}\dot{\varphi}-\dot{\psi}^2\Big)$$
$$+\left(2\dot{t}+2\sqrt{2}kr^2\dot{\varphi}\right)\left(a_2-a_1\dot{t}-\frac{\sqrt{2}a_{10}rk\sin\varphi}{\mathcal{Q}}\right)-\frac{2\left(a_{10}\cos\varphi\mathcal{Q}-a_1\dot{r}\right)\dot{r}}{k^2r^2+1}$$
$$-2\dot{z}\Big(-\frac{1}{2}a_6s-a_1\dot{z}\Big)+\left(a_4-\frac{a_{10}\left(2k^2r^2+1\right)\sin\varphi}{r\mathcal{Q}}-a_1\dot{\varphi}\right)$$
$$\times\left(2\dot{\varphi}\left(2k^2r^4-r^2\mathcal{Q}^2+1\right)\right)+2\sqrt{2}kr^2\dot{t}\Big)-2\dot{\psi}\left(1-a_1\dot{\psi}\right)-z-c\Bigg)=0.$$

(**2**) : For the symmetry operator,

$$a_1\mathbf{X_1}+a_2\mathbf{X_2}+a_4\mathbf{X_4}+\mathbf{X_5}+a_6\mathbf{X_6}+a_9\mathbf{X_9}$$
$$=a_1\frac{\partial}{\partial s}+\left(a_2+a_9\frac{\sqrt{2}rk\cos\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t}+\left(a_9\sin\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r}-a_6\big(\frac{s}{2}\big)\frac{\partial}{\partial z}$$
$$+\left(a_4+a_9\frac{\left(2k^2r^2+1\right)\cos\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi}+\frac{\partial}{\partial\psi},\quad A=z+c.$$

By applying Noether's theorem, the following conservation law is resulted:

$$D_s\Bigg(a_1\left(\dot{t}^2-\frac{\dot{r}^2}{k^2r^2+1}-\dot{z}^2+\left(2k^2r^4-r^2\left(k^2r^2+1\right)\right)\dot{\varphi}^2+2\sqrt{2}kr^2\dot{t}\dot{\varphi}-\dot{\psi}^2\right)$$
$$+\left(2\dot{t}+2\sqrt{2}kr^2\dot{\varphi}\right)\left(a_2-a_1\dot{t}-\frac{\sqrt{2}a_9rk\cos\varphi}{\mathcal{Q}}\right)-\frac{2\left(a_9\sin\varphi\mathcal{Q}-a_1\dot{r}\right)\dot{r}}{\mathcal{Q}^2}$$
$$-2\dot{z}\Big(-\frac{1}{2}a_6s-a_1\dot{z}\Big)+\left(a_4+\frac{a_9\left(2k^2r^2+1\right)\cos\varphi}{r\mathcal{Q}}-a_1\dot{\varphi}\right)$$
$$\times\left(2\dot{\varphi}\left(2k^2r^4-r^2\left(\mathcal{Q}^2+1\right)\right)+2\sqrt{2}kr^2\dot{t}\right)-2\dot{\psi}\left(1-a_1\dot{\psi}\right)-z-c\Bigg)=0.$$

(**3**) : For the symmetry operator

$$\begin{aligned}&a_1\mathbf{X_1}+a_2\mathbf{X_2}+\mathbf{X_3}+a_4\mathbf{X_4}+a_7\mathbf{X_7}+a_9\mathbf{X_9}\\&=a_1\frac{\partial}{\partial s}+\left(a_2+a_9\frac{\sqrt{2}rk\cos\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t}+\left(a_9\sin\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r}+\frac{\partial}{\partial z}\\&\quad+\left(a_4+a_9\frac{\left(2k^2r^2+1\right)\cos\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi}-\left(a_7\frac{s}{2}\right)\frac{\partial}{\partial\psi},\quad A=\psi+c.\end{aligned}$$

By applying Noether's theorem, the following conservation law is resulted:

$$\begin{aligned}D_s\Bigg(&a_1\left(\dot t^2-\frac{\dot r^2}{\mathcal{Q}^2}-\dot z^2+\left(2k^2r^4-r^2\left(\mathcal{Q}^2\right)\right)\dot\varphi^2+2\sqrt{2}kr^2\dot t\dot\varphi-\dot\psi^2\right)\\&+\left(2\dot t+2\sqrt{2}kr^2\dot\varphi\right)\left(a_2-a_1\dot t+\frac{\sqrt{2}a_9rk\cos\varphi}{\mathcal{Q}}\right)-\frac{2\left(a_9\sin\varphi\mathcal{Q}-a_1\dot r\right)\dot r}{\mathcal{Q}^2}\\&-2\dot z\left(1-a_1\dot z\right)+\left(a_4+\frac{a_9\left(2k^2r^2+1\right)\cos\varphi}{r\mathcal{Q}}-a_1\dot\varphi\right)\\&\times\left(2\dot\varphi\left(2k^2r^4-r^2\mathcal{Q}^2\right)+2\sqrt{2}kr^2\dot t\right)+2\dot\psi\left(\frac{1}{2}a_7s+a_1\dot\psi\right)-\psi-c\Bigg)=0.\end{aligned}$$

(**4**) : For the symmetry operator,

$$\begin{aligned}&a_1\mathbf{X_1}+a_2\mathbf{X_2}+\mathbf{X_3}+a_4\mathbf{X_4}+a_7\mathbf{X_7}+a_{10}\mathbf{X_{10}}\\&=a_1\frac{\partial}{\partial s}+\left(a_2-a_{10}\frac{\sqrt{2}rk\sin\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t}+\left(a_{10}\cos\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r}+\frac{\partial}{\partial z}\\&\quad+\left(a_4-a_{10}\frac{\left(2k^2r^2+1\right)\sin\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi}-\left(a_7\frac{s}{2}\right)\frac{\partial}{\partial\psi},\quad A=\psi+c.\end{aligned}$$

By applying Noether's theorem, the following conservation law is resulted:

$$\begin{aligned}D_s\Bigg(&a_1\left(\dot t^2-\frac{\dot r^2}{\mathcal{Q}^2}-\dot z^2+\left(2k^2r^4-r^2\mathcal{Q}^2\right)\dot\varphi^2+2\sqrt{2}kr^2\dot t\dot\varphi-\dot\psi^2\right)\\&+\left(2\dot t+2\sqrt{2}kr^2\dot\varphi\right)\left(a_2-a_1\dot t-\frac{\sqrt{2}a_{10}\ rk\sin\varphi}{\mathcal{Q}}\right)-\frac{2\left(a_{10}\cos\varphi\mathcal{Q}-a_1\dot r\right)\dot r}{\mathcal{Q}^2}\\&-2\dot z\left(1-a_1\dot z\right)+\left(a_4-\frac{a_{10}\left(2k^2r^2+1\right)\sin\varphi}{r\mathcal{Q}}-a_1\dot\varphi\right)\\&\times\left(2\dot\varphi\left(2k^2r^4-r^2\mathcal{Q}^2\right)+2\sqrt{2}kr^2\dot t\right)+2\dot\psi\left(\frac{1}{2}a_7s+a_1\dot\psi\right)-\psi-c\Bigg)=0.\end{aligned}$$

(**5**) : For the symmetry operator,

$$
\begin{aligned}
&a_1\mathbf{X_1} + a_2\mathbf{X_2} + a_4\mathbf{X_4} + \mathbf{X_7} + a_9\mathbf{X_9} \\
&= a_1\frac{\partial}{\partial s} + \left(a_2 + a_9\frac{\sqrt{2}rk\cos\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t} + \left(a_9\sin\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r} + \frac{\partial}{\partial z} \\
&\quad + \left(a_4 + a_9\frac{\left(2k^2r^2+1\right)\cos\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi} - \left(\frac{s}{2}\right)\frac{\partial}{\partial\psi}, \quad A = \psi + c.
\end{aligned}
$$

By applying Noether's theorem, the following conservation law is resulted:

$$
\begin{aligned}
D_s\Bigg(& a_1\left(\dot{t}^2 - \frac{\dot{r}^2}{\mathcal{Q}^2} - \dot{z}^2 + \left(2k^2r^4 - r^2\mathcal{Q}^2\right)\dot{\varphi}^2 + 2\sqrt{2}kr^2\dot{t}\dot{\varphi} - \dot{\psi}^2\right) \\
&+\left(2\dot{t} + 2\sqrt{2}kr^2\dot{\varphi}\right)\left(a_2 - a_1\dot{t} + \frac{\sqrt{2}a_9rk\cos\varphi}{\mathcal{Q}}\right) - \frac{2\left(a_9\sin\varphi\mathcal{Q} - a_1\dot{r}\right)\dot{r}}{\mathcal{Q}^2} \\
&+2a_1\dot{z}^2 + \left(a_4 + \frac{a_9\left(2k^2r^2+1\right)\cos\varphi}{r\mathcal{Q}} - a_1\dot{\varphi}\right) \\
&\times\left(2\dot{\varphi}\left(2k^2r^4 - r^2\mathcal{Q}^2\right) + 2\sqrt{2}kr^2\dot{t}\right) + 2\dot{\psi}\left(\frac{1}{2}s + a_1\dot{\psi}\right) - \psi - c\Bigg) = 0.
\end{aligned}
$$

(**6**) : For the symmetry operator,

$$
\begin{aligned}
&a_1\mathbf{X_1} + a_2\mathbf{X_2} + a_4\mathbf{X_4} + \mathbf{X_7} + a_{10}\mathbf{X_{10}} \\
&= a_1\frac{\partial}{\partial s} + \left(a_2 - a_{10}\frac{\sqrt{2}rk\sin\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t} + \left(a_{10}\cos\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r} + \frac{\partial}{\partial z} \\
&\quad + \left(a_4 - a_{10}\frac{\left(2k^2r^2+1\right)\sin\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi} - \left(\frac{s}{2}\right)\frac{\partial}{\partial\psi}, \quad A = \psi + c.
\end{aligned}
$$

By applying Noether's theorem, the following conservation law is resulted:

$$
\begin{aligned}
D_s\Bigg(& a_1\left(\dot{t}^2 - \frac{\dot{r}^2}{\mathcal{Q}^2} - \dot{z}^2 + \left(2k^2r^4 - r^2\mathcal{Q}^2\right)\right)\dot{\varphi}^2 + 2\sqrt{2}kr^2\dot{t}\dot{\varphi} - \dot{\psi}^2\Big) \\
&+\left(2\dot{t} + 2\sqrt{2}kr^2\dot{\varphi}\right)\left(a_2 - a_1\dot{t} - \frac{\sqrt{2}a_{10}\ rk\sin\varphi}{\mathcal{Q}}\right) - \frac{2\left(a_{10}\mathcal{Q}\cos\varphi - a_1\dot{r}\right)\dot{r}}{\mathcal{Q}^2} \\
&+2\dot{z}^2a_1 + \left(a_4 - \frac{a_{10}\left(2k^2r^2+1\right)\sin\varphi}{r\mathcal{Q}} - a_1\dot{\varphi}\right) \\
&\times\left(2\dot{\varphi}\left(2k^2r^4 - r^2\mathcal{Q}^2\right) + 2\sqrt{2}kr^2\dot{t}\right) + 2\dot{\psi}\left(\frac{1}{2}s + a_1\dot{\psi}\right) - \psi - c\Bigg) = 0.
\end{aligned}
$$

(**7**) : For the symmetry operator,

$$\begin{aligned}&a_1\mathbf{X_1}+a_2\mathbf{X_2}+a_4\mathbf{X_4}+\mathbf{X_6}+a_{10}\mathbf{X_{10}}\\&=a_1\frac{\partial}{\partial s}+\left(a_2-a_{10}\frac{\sqrt{2}rk\sin\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t}+\left(a_{10}\cos\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r}-\left(\frac{s}{2}\right)\frac{\partial}{\partial z}\\&\quad+\left(a_4-a_{10}\frac{\left(2k^2r^2+1\right)\sin\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi}+\frac{\partial}{\partial\psi},\quad A=z+c.\end{aligned}$$

By applying Noether's theorem, the following conservation law is resulted:

$$\begin{aligned}&D_s\Bigg(a_1\left(\dot t^2-\frac{\dot r^2}{\mathcal{Q}^2}-\dot z^2+\left(2k^2r^4-r^2\left(\mathcal{Q}^2\right)\right)\dot\varphi^2+2\sqrt{2}kr^2\dot t\dot\varphi-\dot\psi^2\right)\\&+\left(2\dot t+2\sqrt{2}kr^2\dot\varphi\right)\left(a_2-a_1\dot t-\frac{\sqrt{2}a_{10}\ rk\sin\varphi}{\mathcal{Q}}\right)-\frac{2\left(a_{10}\mathcal{Q}\cos\varphi-a_1\dot r\right)\dot r}{\mathcal{Q}^2}\\&-2\dot z\left(-\frac{1}{2}a_6s-a_1\dot z\right)+\left(a_4-\frac{a_{10}\ \left(2k^2r^2+1\right)\ \sin\varphi}{r\mathcal{Q}}-a_1\dot\varphi\right)\\&\times\left(2\dot\varphi\left(2k^2r^4-r^2\mathcal{Q}\right)+2\sqrt{2}kr^2\dot t\right)+2\dot\psi^2a_1-z-c\Bigg)=0.\end{aligned}$$

(**8**) : For the symmetry operator,

$$\begin{aligned}&a_1\mathbf{X_1}+a_2\mathbf{X_2}+a_4\mathbf{X_4}+\mathbf{X_6}+a_9\mathbf{X_9}\\&=a_1\frac{\partial}{\partial s}+\left(a_2+a_9\frac{\sqrt{2}rk\cos\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t}+\left(a_9\sin\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r}-\left(\frac{s}{2}\right)\frac{\partial}{\partial z}\\&\quad+\left(a_4+a_9\frac{\left(2k^2r^2+1\right)\cos\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi}+\frac{\partial}{\partial\psi},\quad A=z+c.\end{aligned}$$

By applying Noether's theorem, the following conservation law is resulted:

$$\begin{aligned}&D_s\Bigg(a_1\left(\dot t^2-\frac{\dot r^2}{\mathcal{Q}^2}-\dot z^2+\left(2k^2r^4-r^2\mathcal{Q}^2\right)\dot\varphi^2+2\sqrt{2}kr^2\dot t\dot\varphi-\dot\psi^2\right)\\&+\left(2\dot t+2\sqrt{2}kr^2\dot\varphi\right)\left(a_2-a_1\dot t+\frac{\sqrt{2}a_9rk\cos\varphi}{\mathcal{Q}}\right)-\frac{2\left(a_9\mathcal{Q}\sin\varphi-a_1\dot r\right)\dot r}{\mathcal{Q}^2}\\&+2\dot z\left(\frac{1}{2}s+a_1\dot z\right)+\left(a_4+\frac{a_9\left(2k^2r^2+1\right)\cos\varphi}{r\mathcal{Q}}-a_1\dot\varphi\right)\\&\times\left(2\dot\varphi\left(2k^2r^4-r^2\mathcal{Q}^2\right)+2\sqrt{2}kr^2\dot t\right)+2\sqrt{2}kr^2\dot t\Bigg)+2\dot\psi^2a_1-z-c\Bigg)=0.\end{aligned}$$

(**9**) : For the symmetry operator,

$$
\begin{aligned}
&a_1\mathbf{X_1} + a_2\mathbf{X_2} + a_4\mathbf{X_4} + a_8\mathbf{X_8} + \mathbf{X_{10}} \\
&= a_1\frac{\partial}{\partial s} + \left(a_2 - \frac{\sqrt{2}rk\sin\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t} + \left(\cos\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r} - \left(a_8\psi\right)\frac{\partial}{\partial z} \\
&\quad + \left(a_4 - \frac{\left(2k^2r^2+1\right)\sin\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi} + \left(a_8 z\right)\frac{\partial}{\partial\psi}, \quad A = c.
\end{aligned}
$$

By applying Noether's theorem, the following conservation law is resulted:

$$
\begin{aligned}
&D_s\Bigg(a_1\left(\dot{t}^2 - \frac{\dot{r}^2}{\mathcal{Q}^2} - \dot{z}^2 + \left(2k^2r^4 - r^2\mathcal{Q}^2\right)\dot{\varphi}^2 + 2\sqrt{2}kr^2\dot{t}\dot{\varphi} - \dot{\psi}^2\right) \\
&+\left(2\dot{t} + 2\sqrt{2}kr^2\dot{\varphi}\right)\left(a_2 - a_1\dot{t} - \frac{\sqrt{2}rk\sin\varphi}{\mathcal{Q}}\right) - \frac{2\left(cos\varphi\mathcal{Q} - a_1\dot{r}\right)\dot{r}}{\mathcal{Q}^2} \\
&+2\dot{z}\left(a_1\dot{z} + a_8\psi\right) + \left(a_4 - \frac{\left(2k^2r^2+1\right)\sin\varphi}{r\mathcal{Q}} - a_1\dot{\varphi}\right) \\
&\times\left(2\dot{\varphi}\left(2k^2r^4 - r^2\mathcal{Q}^2\right) + 2\sqrt{2}kr^2\dot{t}\right) - 2\dot{\psi}\left(a_8 z - a_1\dot{\psi}\right)\dot{\psi} - c\Bigg) = 0.
\end{aligned}
$$

(**10**) : For the symmetry operator,

$$
\begin{aligned}
&a_1\mathbf{X_1} + a_2\mathbf{X_2} + a_4\mathbf{X_4} + a_8\mathbf{X_8} + a_9\mathbf{X_9} \\
&= a_1\frac{\partial}{\partial s} + \left(a_2 + a_9\frac{\sqrt{2}rk\cos\varphi}{\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial t} + \left(a_9\sin\varphi\sqrt{k^2r^2+1}\right)\frac{\partial}{\partial r} - \left(a_8\psi\right)\frac{\partial}{\partial z} \\
&\quad + \left(a_4 + a_9\frac{\left(2k^2r^2+1\right)\cos\varphi}{r\sqrt{k^2r^2+1}}\right)\frac{\partial}{\partial\varphi} + \left(a_8 z\right)\frac{\partial}{\partial\psi}, \quad A = c.
\end{aligned}
$$

By applying Noether's theorem, the following conservation law is resulted:

$$
\begin{aligned}
&D_s\Bigg(a_1\left(\dot{t}^2 - \frac{\dot{r}^2}{\mathcal{Q}^2} - \dot{z}^2 + \left(2k^2r^4 - r^2\mathcal{Q}^2\right)\dot{\varphi}^2 + 2\sqrt{2}kr^2\dot{t}\dot{\varphi} - \dot{\psi}^2\right) \\
&+\left(2\dot{t} + 2\sqrt{2}kr^2\dot{\varphi}\right)\left(a_2 - a_1\dot{t} + \frac{\sqrt{2}a_9rk\cos\varphi}{\mathcal{Q}}\right) - \frac{2\left(a_9\mathcal{Q}\sin\varphi - a_1\dot{r}\right)\dot{r}}{\mathcal{Q}^2} \\
&+2\dot{z}\left(a_8\psi + a_1\dot{z}\right) + \left(a_4 + \frac{a_9\left(2k^2r^2+1\right)\cos\varphi}{r\mathcal{Q}} - a_1\dot{\varphi}\right) \\
&\times\left(2\dot{\varphi}\left(2k^2r^4 - r^2\mathcal{Q}^2\right) + 2\sqrt{2}kr^2\dot{t}\right) - 2\dot{\psi}\left(a_8 z - a_1\dot{\psi}\right) - c\Bigg) = 0.
\end{aligned}
$$

Conclusion

In recent years, inquiring into rotating fluids in the context of general relativity has received noteworthy consideration principally after Godel proposed relativistic model of a rotating dust universe. The noticeable point is that stationary Kaluza–Klein perfect fluid models in standard Einstein theory are not available in literature. Consequently, obtaining and analyzing such solutions in order to investigate the influences of dimensionality on the distinct physical parameters is of special significance. Tikekar and Patel [31] have formulated the Kaluza–Klein field equations for cylindrically symmetric rotating distributions of perfect fluid. They have reported a set of physically worthwhile solutions, which is believed to be the first such Kaluza–Klein solutions and it includes the Kaluza–Klein counterpart of Davidson's solution. In this paper, we have comprehensively analyzed the problem of symmetries and conservation laws for some specific solutions of Kaluza–Klein field equations for stationary symmetric fluid models in standard Einstein theory. For this purpose, we have considered a physically viable stationary Kaluza–Klein perfect fluid solution, which is reported in [31]. The general theory of relativity, which can be regarded as the field theory of gravitation is fundamentally governed by the Einstein field equations. These equations are extremely nonlinear and are demonstrated in terms of the Lorentzian metric g_{ab}. Taking into account this nonlinearity, obtaining their exact solutions is totally difficult. Hence, it has been one of the basic problems in general relativity to analyze the solutions of the Einstein field equations by means of the symmetries they possess. According to the significance of these symmetries in description of the physics of the gravitational fields, they have been extensively investigated and a large body of literature is available on them until now. From the geometric point of view, symmetries are so fruitful mainly due to the fact that they are directly connected to the conservation laws via Noether's theorem. Since the geodesic equations follow from the variation of the geodesic Lagrangian defined by the metric, the vector fields that leave the action integral invariant, namely, Noether symmetries yield conservation laws. Moreover, from the geometric approach the symmetries of a manifold are described by its Killing vectors or isometries, which form a Lie algebra structure as well. In other words, the symmetries of the manifold are mainly inherited by the geodesic equations on it with additional symmetries. They yield quantities that are conserved under geodesic motion and result in first integrals of the geodesic equations. In the current research, first of all, for this specific solution, by considering the Lagrangian that is determined directly from the metric, we have computed the corresponding geodesic equations as the Euler Lagrange equations. Secondly, we have obtained the point generators of the one parameter Lie groups of transformations that leave invariant the action integral corresponding to the Lagrangian, viz., Noether symmetries. Moreover, a brief discussion regarding the algebraic structure of the resulted Lie algebra of Noether symmetries is presented. In addition, a complete classification of symmetry subalgebras for the system of geodesic equations is proposed. For this purpose, the adjoint representation is applied in order to construct an optimal system of one-dimensional subalgebras, which provides the preliminary classification of group invariant solutions for the system of geodesic equations. Particularly, by re-expressing the analyzed metric in the orthogonal coframe, the corresponding Killing vector fields, which can be regarded as one of the most significant types of symmetries and are considered as the smooth vector fields, which preserve the metric tensor are thoroughly calculated. Significantly, for the resulted Lie algebra of Killing vector fields, the associated basis for the original Lie algebra is determined in which the Lie algebras will be appropriately decomposed into an internal direct sum of subalgebras, where each summand is indecomposable. Principally, an entire set of conservation laws for our stationary Kaluza–Klein perfect fluid solution is computed via the celebrated Noether's theorem, which is fundamentally relied on the geodesic Lagrangian and the corresponding Noether symmetries, which leave the action integral invariant.

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Department of Mathematics, Faculty of Mathematical Sciences, Alzahra University, Tehran, P.O.Box 1993893973, Iran.

Email address: f.ahangari@alzahra.ac.ir; fa.ahangari@gmail.com