Khayyam J. Math. 10 (2024), no. 1, 108-125
DOI: 10.22034/KJM.2024.392808.2836

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

THE η-HERMITIAN SOLUTIONS OF SOME QUATERNION MATRIX EQUATIONS

RADJA BELKHIRI[1] AND SIHEM GUERARRA[2*]

Communicated by B. Kuzma

Abstract. Let $\mathbb{H}^{n\times m}$ be the set of all $n\times m$ matrices over the real quaternion algebra. In this paper, we derive the solvability conditions for the common η-Hermitian solution to the system of two quaternion matrix equations $A_1X_1A_1^{\eta*}+B_1Y_1B_1^{\eta*}=C_1$ and $A_2X_2A_2^{\eta*}+B_2Y_2B_2^{\eta*}=C_2$. As applications, we obtain necessary and sufficient conditions for the pair of quaternion matrix equations $A_1X_1A_1^{\eta*}=C_1$ and $A_2X_2A_2^{\eta*}=C_2$ to have common η-Hermitian solution. In additions, we establish formulas of the extremal ranks of the quaternion η-Hermitian matrix expression $A_2X_2A_2^{\eta*}=C_2$ with respect to η-Hermitian solution of $A_1X_1A_1^{\eta*}=C_1$, then we derive extremal ranks of the generalized η-Hermitian Schur complement $S_{A_1}=D-B^{\eta*}A_1^{-}B$ with respect to η-Hermitian generalized inverse A_1^{-} of A_1, which is a solution to the quaternion matrix equation $A_1X_1A_1^{\eta*}=C_1$.

1. Introduction and preliminaries

Throughout this paper, $\mathbb{R}$ and $\mathbb{C}$ stand for the real number field and the complex number field, respectively. Let $\mathbb{H}^{m\times n}$ be the set of $m\times n$ matrices over the real quaternion Algebra:

$$\mathbb{H}=\left\{a_0+a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k}\middle|\ \boldsymbol{i}^2=\boldsymbol{j}^2=\boldsymbol{ijk}=-1,\ a_0,a_1,a_2,a_3\in\mathbb{R}\right\}.$$

The symbols, A^* and $r(A)$ stand for the conjugate transpose and the rank of A, respectively. Also, I_n denotes the identity matrix of order n. The Moore–Penrose

Date: Received: 11 April 2023; Revised: 24 February 2024; Accepted: 25 February 2024.
* Corresponding author.
2020 *Mathematics Subject Classification.* Primary 15A24; Secondary 11R52, 15A09, 15B57.
Key words and phrases. System of quaternion matrix equations, Moore-Penrose inverse, η-Hermitian solution, Schur Complement.

generalized inverse of a given matrix $A \in \mathbb{H}^{m\times n}$ is defined to be the unique matrix symbolized by A^+ and satisfying the following four matrix equations:

$$\text{(a) } AXA = A, \quad \text{(b) } XAX = X, \quad \text{(c) } (AX)^* = AX, \quad \text{(d) } (XA)^* = XA.$$

The Moore–Penrose inverse has been the subject of many researches (see [1, 7]). Furthermore, L_A and R_A stand for the two projectors $L_A = I_n - A^+A$ and $R_A = I_m - AA^+$ induced by $A \in \mathbb{H}^{m\times n}$.

A square matrix A is called an η-Hermitian matrix if $A = A^{\eta*} = -\eta A^*\eta$, where $\eta \in \{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$. The notion of η-Hermitian quaternion matrices was first studied by Took, Mandic and Zhang [8] in 2011. There have been some papers to discuss the topics related to η-Hermitian quaternion matrix (see [9, 4, 11]). For instance, He and Wang [2] provided some necessary and sufficient conditions for the existence of solution to the quaternion matrix equation

$$A_1X + (A_1X)^{\eta*} + B_1YB_1^{\eta*} + C_1ZC_1^{\eta*} = D_1,$$

where Y and Z are required to be η-Hermitian matrices. As applications, they derived necessary and sufficient conditions for the two quaternion matrix equations:

$$A_1X_1A_1^{\eta*} = C_1, \tag{1.1}$$

$$A_1X_1A_1^{\eta*} + B_1Y_1B_1^{\eta*} = C_2. \tag{1.2}$$

to have η-Hermitian solutions. They also presented the general solutions to (1.1) and (1.2) when they are consistent.

In 2006, Liu [5] gave the solvability conditions to the system of quaternion matrix equations with two unknowns

$$A_1X_1 + Y_1B_1 = C_1,$$
$$A_2X_2 + Y_2B_2 = C_2.$$

Yu [10] derived extremal ranks of Schur Complement subject to system of quaternion matrix equations

$$A_1X = C_1,$$
$$XB_1 = C_2.$$

Motivated by the works mentioned above, this paper is organized as follows. In section 2, we consider the common η-Hermitian solution to the system of quaternion matrix equations:

$$\begin{cases} A_1X_1A_1^{\eta*} + B_1Y_1B_1^{\eta*} = C_1, \\ A_2X_2A_2^{\eta*} + B_2Y_2B_2^{\eta*} = C_2, \end{cases} \tag{1.3}$$

where $C_i = C_i^{\eta*} \in \mathbb{H}^{m_i\times m_i}$, $A_i \in \mathbb{H}^{m_i\times n}$, and $B_i \in \mathbb{H}^{m_i\times k}$ $(i = 1,\ 2)$ are given and $X_i = X_i^{\eta*} \in \mathbb{H}^{n\times n}$ and $Y_i = Y_i^{\eta*} \in \mathbb{H}^{k\times k}$ are the unknown matrices. Also, we derive the solvability conditions for the system of quaternion matrix equations:

$$\begin{cases} A_1X_1A_1^{\eta*} = C_1, \\ A_2X_2A_2^{\eta*} = C_2, \end{cases} \tag{1.4}$$

where $C_i = C_i^{\eta*} \in \mathbb{H}^{m_i \times m_i}$ and $A_i \in \mathbb{H}^{m_i \times n}$ $(i = 1,\, 2)$ are given, and $X_i = X_i^{\eta*} \in \mathbb{H}^{n\times n}(i = 1,\, 2)$ are unknown. In section 3, we first derive extremal ranks of the quaternion matrix expression $f(X) = C_2 - A_2X_1A_2^{\eta*}$ with respect to η-Hermitian solution of the quaternion matrix equation (1.1). As an application, we establish maximal and minimal ranks of the generalized η-Hermitian Schur complement $S_{A_1} = D - B^{\eta*}A_1^{-}B$ with respect to η-Hermitian generalized inverse A_1^{-} of A_1, which is a solution to the quaternion matrix equation (1.1).

The following lemma is due to Marsagalia and Styan [7], which can be easily generalized to $\mathbb{H}$.

Lemma 1.1. *Let $A \in \mathbb{H}^{m\times n}$, $B \in \mathbb{H}^{m\times k}$, $C \in \mathbb{H}^{l\times n}$, $D \in \mathbb{H}^{m\times p}$, $Q \in \mathbb{H}^{m_1\times k}$, and $P \in \mathbb{H}^{l\times n_1}$ be given. Then*

$$r\begin{bmatrix} A & B \end{bmatrix} = r(B) - r(R_BA) = r(A) - r(R_AB),$$
$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) - r(CL_A) = r(C) - r(AL_C),$$
$$r\begin{bmatrix} A & BL_Q \\ R_PC & 0 \end{bmatrix} = r\begin{bmatrix} A & B & 0 \\ C & 0 & P \\ 0 & Q & 0 \end{bmatrix} - r(P) - r(Q).$$

Some important properties of η-Hermitian matrix are given in the following lemma.

Lemma 1.2. [2] *Let $A \in \mathbb{H}^{m\times n}$ be given. Then*

$$\left(A^{\eta*}\right)^{+} = \left(A^{+}\right)^{\eta*},$$
$$r\left(A^{\eta*}\right) = r(A),$$
$$\left(A^{+}A\right)^{\eta*} = A^{\eta*}\left(A^{+}\right)^{\eta*},$$
$$\left(AA^{+}\right)^{\eta*} = \left(A^{+}\right)^{\eta*}A^{\eta*},$$
$$\left(L_A\right)^{\eta*} = R_{A^{\eta*}},$$
$$\left(R_A\right)^{\eta*} = L_{A^{\eta*}}.$$

In order to establish the solvability conditions for the η-Hermitian solution to system (1.3), we need the following results on η-Hermitian solution of the matrix equation (1.2).

Lemma 1.3. [2] *Let A_1, B_1 and $C_1 = C_1^{\eta*}$ be given. Set $M = R_{A_1}B_1$ and $S = B_1L_M$. Then the following statements are equivalent:*
(1) Matrix equation (1.2) has a pair of η-Hermitian solutions X_1 and Y_1.
(2)

$$R_MR_{A_1}C_1 = 0,\; R_{A_1}C_1\left(R_{B_1}\right)^{\eta*} = 0.$$

(3)

$$r\begin{bmatrix} A_1 & C_1 \\ 0 & B_1^{\eta*} \end{bmatrix} = r(A_1) + r(B_1),\; r\begin{bmatrix} A_1 & B_1 & C_1 \end{bmatrix} = r\begin{bmatrix} A_1 & B_1 \end{bmatrix}.$$

In this case, the η-Hermitian solution to matrix equation (1.2) can be expressed as

$$\begin{aligned} X_1 =& A_1^+ C_1 \left(A_1^+\right)^{\eta*} - \frac{1}{2} A_1^+ B_1 M^+ C_1 \left[I + \left(B_1^+\right)^{\eta*} S^{\eta*}\right] \left(A_1^+\right)^{\eta*} \\ & - \frac{1}{2} A_1^+ \left(I + S B_1^+\right) C_1 \left(M^+\right)^{\eta*} B_1^{\eta*} \left(A_1^+\right)^{\eta*} \\ & - A_1^+ S W_2 S^{\eta*} \left(A_1^+\right)^{\eta*} + L_{A_1} U + U^{\eta*} \left(L_{A_1}\right)^{\eta}, \\ Y_1 =& \frac{1}{2} M^+ C_1 \left(B_1^+\right)^{\eta*} \left[I + \left(S^+ S\right)^{\eta}\right] + \frac{1}{2} \left(I + S^+ S\right) B_1^+ C_1 \left(M^+\right)^{\eta*} \\ & + L_M W_2 \left(L_M\right)^{\eta} + V L_{B_1}^{\eta} + L_{B_1} V^{\eta*} \\ & + L_M L_S W_1 + W_1^{\eta*} \left(L_S\right)^{\eta} \left(L_M\right)^{\eta}, \end{aligned}$$

where W_1, U, V and $W_2 = W_2^{\eta*}$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Lemma 1.4. *Let $A_1 \in \mathbb{H}^{m\times n}$ and $C_1 = C_1^{\eta*} \in \mathbb{H}^{m\times m}$ be given. Then the real quaternion matrix equation (1.1) has an η-Hermitian solution if and only if $A_1 A_1^+ C_1 = C_1$, that is, $r\begin{bmatrix} A_1 & C_1 \end{bmatrix} = r(A_1)$. In this case, the η-Hermitian solution of can be expressed as*

$$X = A_1^+ C_1 \left(A_1^+\right)^{\eta*} + L_{A_1} U + U^{\eta*} \left(L_{A_1}\right)^{\eta*},$$

where U is an arbitrary matrix over $\mathbb{H}$ with appropriate size.

Khan, Wang, and Song [3] derived the minimal ranks of the following quaternion matrix expression:

$$f\left(U_1, W_1\right) = A_1 - B_1 U_1 - \left(B_1 U_1\right)^{(*)} - C_1 W_1 C_1^{(*)}, \tag{1.5}$$

where $A_1 = A_1^{(*)}$ and $W_1 = W_1^{(*)}$.

He and Wang [2] derived the minimal rank of the matrix expression

$$P\left(U_1, W_1\right) = A_1 - B_1 U_1 - \left(B_1 U_1\right)^{\eta*} - C_1 W_1 C^{\eta*}, \tag{1.6}$$

by similar approach in [3].

Lemma 1.5. [2] *Let $P\left(U_1, W_1\right)$ be as given in (1.6) with $A = A^{\eta*}$. Then*

$$\min_{U, W = W^{\eta*}} r\left[P\left(U_1, W_1\right)\right] = 2r\begin{bmatrix} A & B & C \\ B^{\eta*} & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B & C \\ B^{\eta*} & 0 & 0 \\ C^{\eta*} & 0 & 0 \end{bmatrix} - 2r(B). \tag{1.7}$$

Liu and Tian [6] derived the maximal and minimal ranks of the matrix expression $A - BXC - (BXC)^*$ over the complex field $\mathbb{C}$. We can obtain the maximal and minimal ranks of the matrix expression $A - BXC - (BXC)^{\eta*}$ over the quaternion algebra.

Lemma 1.6. [6]*Let $A = A^{\eta*} \in \mathbb{H}^{m\times m}$, $B \in \mathbb{H}^{m\times n}$, and $C \in \mathbb{H}^{p\times m}$ be given. If $R(B) \subseteq R\left(C^{\eta*}\right)$, then*

$$\max_{X\in\mathbb{H}^{p\times n}} r\left[A - BXC - (BXC)^{\eta*}\right] = \min\left\{r\begin{bmatrix} A & C^{\eta*} \end{bmatrix}, r\begin{bmatrix} A & B \\ B^{\eta*} & 0 \end{bmatrix}\right\}, \tag{1.8}$$

$$\min_{X\in\mathbb{H}^{p\times n}} r\left[A-BXC-(BXC)^{\eta*}\right]=2r\left[\begin{array}{cc} A & C^{\eta*}\end{array}\right]+r\left[\begin{array}{cc} A & B \\ B^{\eta*} & 0\end{array}\right]-2r\left[\begin{array}{cc} A & B \\ C & 0\end{array}\right]. \tag{1.9}$$

2. The common η-Hermitian solution of the system of quaternion matrix equations (1.3)

The goal of this section is to derive necessary and sufficient conditions for the system of quaternion matrix equations (1.3) to have common η-Hermitian solution. Now, we give the fundamental result of this section.

Theorem 2.1. *Let $A_i\in\mathbb{H}^{m_i\times n}$,$B_i\in\mathbb{H}^{m_i\times k}$, and $C_i=C_i^{\eta*}\in\mathbb{H}^{m_i\times m_i}$ $(i=1,\ 2)$ be given, and assume that the pair of quaternion matrix equations in (1.3) has an η-Hermitian solution. We put*

$$M_i=R_{A_i}B_i,\ S_i=B_iL_{M_i}\qquad \text{for } (i=1,\ 2).$$

Denote

$$D_1=\left[\begin{array}{ccccc} 0 & 0 & 0 & A_1^{\eta*} & A_2^{\eta*} \\ A_1 & B_1 & 0 & C_1 & 0 \\ A_2 & 0 & B_2 & 0 & -C_2\end{array}\right],$$

$$D_2=\left[\begin{array}{ccccccc} B_1^{\eta*} & B_2^{\eta*} & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & -B_1 & 0 & 0 & 0 & 0 \\ 0 & C_2 & B_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & 0 & A_1 & 0 \\ 0 & 0 & 0 & 0 & B_2 & 0 & A_2\end{array}\right],$$

$$L_1=\left[\begin{array}{ccccc} 0 & 0 & 0 & A_1^{\eta*} & -A_2^{\eta*} \\ 0 & 0 & 0 & B_1^{\eta*} & 0 \\ 0 & 0 & 0 & 0 & B_2^{\eta*} \\ A_1 & B_1 & 0 & C_1 & 0 \\ A_2 & 0 & B_2 & 0 & C_2\end{array}\right],$$

$$L_2=\left[\begin{array}{ccccccccc} 0 & 0 & 0 & 0 & B_2^{\eta*} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1^{\eta*} & 0 & 0 & B_2^{\eta*} & 0 & 0 \\ 0 & 0 & -B_1 & C_1 & 0 & 0 & 0 & 0 & 0 \\ B_2 & 0 & 0 & 0 & C_2 & 0 & 0 & 0 & 0 \\ 0 & B_1 & 0 & 0 & 0 & 0 & 0 & A_1 & 0 \\ 0 & 0 & B_2 & 0 & 0 & A_1^{\eta*} & 0 & 0 & A_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_2^{\eta*} & 0 & 0\end{array}\right].$$

Then,

$$\min_{\substack{A_1X_1A_1^{\eta*}+B_1Y_1B_1^{\eta*}=C_1\\ A_2X_2A_2^{\eta*}+B_2Y_2B_2^{\eta*}=C_2}} r\left(X_1-X_2\right)=2r\left(D_1\right)-r\left(L_1\right)-2r\left[\begin{array}{c} A_1 \\ A_2\end{array}\right], \tag{2.1}$$

$$\min_{\substack{A_1X_1A_1^{\eta*}+B_1Y_1B_1^{\eta*}=C_1\\ A_2X_2A_2^{\eta*}+B_2Y_2B_2^{\eta*}=C_2}} r\left(Y_1-Y_2\right)=2r\left(D_2\right)-r\left(L_1\right)-2r\left[\begin{array}{c} B_1 \\ B_2\end{array}\right]. \tag{2.2}$$

Proof. It follows from Lemma 1.3 that the general η-Hermitian solution to quaternion matrix equation $A_iX_iA_i^{\eta*}+B_iY_iB_i^{\eta*}=C_i$ $(i=1,\,2)$ can be written as

$$\begin{aligned}X_1=&A_1^{+}C_1\left(A_1^{+}\right)^{\eta*}-\frac{1}{2}A_1^{+}B_1M_1^{+}C_1\left[I+\left(B_1^{+}\right)^{\eta*}S_1^{\eta*}\right]\left(A_1^{+}\right)^{\eta*}\\&-\frac{1}{2}A_1^{+}\left(I+S_1B_1^{+}\right)C_1\left(M_1^{+}\right)^{\eta*}B_1^{\eta*}\left(A_1^{+}\right)^{\eta*}\\&-A_1^{+}S_1W_2S_1^{\eta*}\left(A_1^{+}\right)^{\eta*}+L_{A_1}U_1+U_1^{\eta*}\left(L_{A_1}\right)^{\eta}\\:=&X_{01}-A_1^{+}S_1W_2S_1^{\eta*}\left(A_1^{+}\right)^{\eta*}+L_{A_1}U_1+U_1^{\eta*}\left(L_{A_1}\right)^{\eta},\end{aligned}$$

$$\begin{aligned}X_2=&A_2^{+}C_2\left(A_2^{+}\right)^{\eta*}-\frac{1}{2}A_2^{+}B_2M_2^{+}C_2\left[I+\left(B_2^{+}\right)^{\eta*}S_2^{\eta*}\right]\left(A_2^{+}\right)^{\eta*}\\&-\frac{1}{2}A_2^{+}\left(I+S_2B_2^{+}\right)C_2\left(M_2^{+}\right)^{\eta*}B_2^{\eta*}\left(A_2^{+}\right)^{\eta*}\\&-A_2^{+}S_2W_2'S_2^{\eta*}\left(A_2^{+}\right)^{\eta*}+L_{A_2}U_2+U_2^{\eta*}\left(L_{A_2}\right)^{\eta}\\:=&X_{02}-A_2^{+}S_2W_2'S_2^{\eta*}\left(A_2^{+}\right)^{\eta*}+L_{A_2}U_2+U_2^{\eta*}\left(L_{A_2}\right)^{\eta},\end{aligned}$$

$$\begin{aligned}Y_1=&\frac{1}{2}M_1^{+}C_1\left(B_1^{+}\right)^{\eta*}\left[I+\left(S_1^{+}S_1\right)^{\eta}\right]+\frac{1}{2}\left(I+S_1^{+}S_1\right)B_1^{+}C_1\left(M_1^{+}\right)^{\eta*}\\&+L_{M_1}W_2\left(L_{M_1}\right)^{\eta}+V_1L_{B_1}^{\eta}+L_{B_1}V_1^{\eta*}\\&+L_{M_1}L_{S_1}W_1+W_1^{\eta*}\left(L_{S_1}\right)^{\eta}\left(L_{M_1}\right)^{\eta}\\:=&Y_{01}+L_{M_1}W_2\left(L_{M_1}\right)^{\eta}+V_1L_{B_1}^{\eta}+L_{B_1}V_1^{\eta*}+L_{M_1}L_{S_1}W_1+W_1^{\eta*}\left(L_{S_1}\right)^{\eta}\left(L_{M_1}\right)^{\eta},\end{aligned}$$

$$\begin{aligned}Y_2=&\frac{1}{2}M_2^{+}C_2\left(B_2^{+}\right)^{\eta*}\left[I+\left(S_2^{+}S_2\right)^{\eta}\right]+\frac{1}{2}\left(I+S_2^{+}S_2\right)B_2^{+}C_2\left(M_2^{+}\right)^{\eta*}\\&+L_{M_2}W_2'\left(L_{M_2}\right)^{\eta}+V_2L_{B_2}^{\eta}+L_{B_2}V_2^{\eta*}\\&+L_{M_2}L_{S_2}W_1'+W_1'^{\eta*}\left(L_{S_2}\right)^{\eta}\left(L_{M_2}\right)^{\eta}\\:=&Y_{02}+L_{M_2}W_2'\left(L_{M_2}\right)^{\eta}+V_2L_{B_2}^{\eta}+L_{B_2}V_2^{\eta*}+L_{M_2}L_{S_2}W_1'+W_1'^{\eta*}\left(L_{S_2}\right)^{\eta}\left(L_{M_2}\right)^{\eta},\end{aligned}$$

where X_{0i} and Y_{0i} are special η-Hermitian solutions to $A_iX_iA_i^{\eta*}+B_iY_iB_i^{\eta*}=C_i$ for $(i=1,\,2)$ and $U_1,\ V_1,\ U_2,\ V_2,\ W_1,\ W_1',\ W_2=W_2^{\eta*}$, and $W_2'=W_2'^{\eta*}$ are arbitrary matrices with appropriate sizes.
Thus, the differences X_1-X_2 and Y_1-Y_2 can be written as

$$\begin{aligned}X_1-X_2=&X_{01}-X_{02}+\begin{bmatrix}A_1^{+}S_1 & A_2^{+}S_2\end{bmatrix}\begin{bmatrix}-W_2 & 0\\ 0 & W_2'\end{bmatrix}\begin{bmatrix}S_1^{\eta*}\left(A_1^{+}\right)^{\eta*}\\ S_2^{\eta*}\left(A_2^{+}\right)^{\eta*}\end{bmatrix}\\&+\begin{bmatrix}L_{A_1} & L_{A_2}\end{bmatrix}\begin{bmatrix}U_1\\ -U_2\end{bmatrix}+\begin{bmatrix}U_1^{\eta*} & -U_2^{\eta*}\end{bmatrix}\begin{bmatrix}\left(L_{A_1}\right)^{\eta}\\ \left(L_{A_2}\right)^{\eta}\end{bmatrix}\\=&X_{01}-X_{02}+N_1U+\left(N_1U\right)^{\eta*}+P_1WP_1^{\eta*},\end{aligned}\tag{2.3}$$

$$Y_1-Y_2=Y_{01}-Y_{02}+\begin{bmatrix}L_{M_1} & L_{M_2}\end{bmatrix}\begin{bmatrix}W_2 & 0\\ 0 & -W_2'\end{bmatrix}\begin{bmatrix}\left(L_{M_1}\right)^{\eta}\\ \left(L_{M_2}\right)^{\eta}\end{bmatrix}$$

$$+\begin{bmatrix} V_1 & -V_2 & W_1^{\eta*} & -W_1'^{\eta*} \end{bmatrix}\begin{bmatrix} L_{B_1}^{\eta} \\ L_{B_2}^{\eta} \\ (L_{S_1})^{\eta}(L_{M_1})^{\eta} \\ (L_{S_2})^{\eta}(L_{M_2})^{\eta} \end{bmatrix}$$

$$+\begin{bmatrix} L_{B_1} & L_{B_2} & L_{M_1}L_{S_1} & L_{M_2}L_{S_2} \end{bmatrix}\begin{bmatrix} V_1^{\eta*} \\ -V_2^{\eta*} \\ W_1 \\ -W_1' \end{bmatrix}$$

$$= Y_{01} - Y_{02} + N_2U' + (N_2U')^{\eta*} + P_2W'P_2^{\eta*}, \tag{2.4}$$

where $N_1 = \begin{bmatrix} L_{A_1} & L_{A_2} \end{bmatrix}$, $P_1 = \begin{bmatrix} A_1^{+}S_1 & A_2^{+}S_2 \end{bmatrix}$,
$N_2 = \begin{bmatrix} L_{B_1} & L_{B_2} & L_{M_1}L_{S_1} & L_{M_2}L_{S_2} \end{bmatrix}$, and $P_2 = \begin{bmatrix} L_{M_1} & L_{M_2} \end{bmatrix}$.
Applying (1.7) to (2.3) and (2.4), we obtain

$$\min_{\substack{A_1X_1A_1^{\eta*}+B_1Y_1B_1^{\eta*}=C_1 \\ A_2X_2A_2^{\eta*}+B_2Y_2B_2^{\eta*}=C_2}} r(X_1 - X_2) = 2r\begin{bmatrix} X_{01}-X_{02} & N_1 & P_1 \\ N_1^{\eta*} & 0 & 0 \end{bmatrix} - r\begin{bmatrix} X_{01}-X_{02} & N_1 & P_1 \\ N_1^{\eta*} & 0 & 0 \\ P_1^{\eta*} & 0 & 0 \end{bmatrix} - 2r(N_1). \tag{2.5}$$

$$\min_{\substack{A_1X_1A_1^{\eta*}+B_1Y_1B_1^{\eta*}=C_1 \\ A_2X_2A_2^{\eta*}+B_2Y_2B_2^{\eta*}=C_2}} r(Y_1 - Y_2) = 2r\begin{bmatrix} Y_{01}-Y_{02} & N_2 & P_2 \\ N_2^{\eta*} & 0 & 0 \end{bmatrix} - r\begin{bmatrix} Y_{01}-Y_{02} & N_2 & P_2 \\ N_2^{\eta} & 0 & 0 \\ P_2^{\eta*} & 0 & 0 \end{bmatrix} - 2r(N_2). \tag{2.6}$$

Applying Lemma 1.1, bloc Gaussian eliminations and simplifying by $A_1A_1^{+}B_1L_{M_1} = B_1L_{M_1}$, $R_{M_1^{\eta*}}B_1^{\eta*}\left(A_1^{+}\right)^{\eta*}A_1^{\eta*} = R_{M_1^{\eta*}}B_1^{\eta*}$, and $A_iX_{0i}A_i^{\eta*} + B_iY_{0i}B_i^{\eta*} = C_i$ for $(i = 1, 2)$, we obtain

$$r\begin{bmatrix} X_{01}-X_{02} & N_1 & P_1 \\ N_1^{\eta*} & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} X_{01}-X_{02} & L_{A_1} & L_{A_2} & A_1^{+}S_1 & A_2^{+}S_2 \\ (L_{A_1})^{\eta} & 0 & 0 & 0 & 0 \\ (L_{A_2})^{\eta} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} X_{01}-X_{02} & L_{A_1} & L_{A_2} & A_1^{+}S_1 & A_2^{+}S_2 \\ R_{A_1^{\eta*}} & 0 & 0 & 0 & 0 \\ R_{A_2^{\eta*}} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} X_{01}-X_{02} & I_n & I_n & A_1^{+}S_1 & A_2^{+}S_2 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & A_1^{\eta*} & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & A_2^{\eta*} \\ 0 & A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$
\begin{aligned}
&- 2r(A_1) - 2r(A_2) \\
&= r\begin{bmatrix} 0 & I_n & I_n & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & A_1^{\eta*} & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & A_2^{\eta*} \\ -A_1X_{01} & A_1 & 0 & -A_1A_1^{+}B_1L_{M_1} & 0 & 0 & 0 \\ A_2X_{02} & 0 & A_2 & 0 & -A_2A_2^{+}B_2L_{M_2} & 0 & 0 \end{bmatrix} \\
&- 2r(A_1) - 2r(A_2) \\
&= r\begin{bmatrix} 0 & I_n & 0 & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -A_1^{\eta*} & A_2^{\eta*} \\ 0 & 0 & -A_1 & -B_1L_{M_1} & 0 & A_1X_{01}A_1^{\eta*} & 0 \\ 0 & 0 & A_2 & 0 & -B_2L_{M_2} & 0 & -A_2X_{02}A_2^{\eta*} \end{bmatrix} \\
&- 2r(A_1) - 2r(A_2) \\
&= 2n + r\begin{bmatrix} 0 & 0 & 0 & -A_1^{\eta*} & A_2^{\eta*} \\ -A_1 & -B_1 & 0 & C_1 & 0 \\ A_2 & 0 & -B_2 & 0 & -C_2 \\ 0 & M_1 & 0 & 0 & 0 \\ 0 & 0 & M_2 & 0 & 0 \end{bmatrix} \\
&- 2r(A_1) - 2r(A_2) - r(M_1) - r(M_2) \\
&= 2n + r\begin{bmatrix} 0 & 0 & 0 & -A_1^{\eta*} & A_2^{\eta*} & 0 & 0 \\ -A_1 & -B_1 & 0 & C_1 & 0 & 0 & 0 \\ A_2 & 0 & -B_2 & 0 & -C_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_2 \end{bmatrix} \\
&- 2r(A_1) - 2r(A_2) - r\begin{bmatrix} B_1 & A_1 \end{bmatrix} - r\begin{bmatrix} B_2 & A_2 \end{bmatrix} \\
&= 2n + r\begin{bmatrix} 0 & 0 & 0 & A_1^{\eta*} & A_2^{\eta*} \\ A_1 & B_1 & 0 & C_1 & 0 \\ A_2 & 0 & B_2 & 0 & -C_2 \end{bmatrix} - r(A_1) - r(A_2) \\
&- r\begin{bmatrix} B_1 & A_1 \end{bmatrix} - r\begin{bmatrix} B_2 & A_2 \end{bmatrix} \\
&= 2n + r(D_1) - r(A_1) - r(A_2) - r\begin{bmatrix} B_1 & A_1 \end{bmatrix} - r\begin{bmatrix} B_2 & A_2 \end{bmatrix}. \qquad (2.7)
\end{aligned}
$$

$$
\begin{aligned}
& r\begin{bmatrix} X_{01} - X_{02} & N_1 & P_1 \\ N_1^{\eta*} & 0 & 0 \\ P_1^{\eta*} & 0 & 0 \end{bmatrix} \\
&= r\begin{bmatrix} X_{01} - X_{02} & L_{A_1} & L_{A_2} & A_1^{+}S_1 & A_2^{+}S_2 \\ (L_{A_1})^{\eta} & 0 & 0 & 0 & 0 \\ (L_{A_2})^{\eta} & 0 & 0 & 0 & 0 \\ S_1^{\eta*}\left(A_1^{+}\right)^{\eta*} & 0 & 0 & 0 & 0 \\ S_2^{\eta*}\left(A_2^{+}\right)^{\eta*} & 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&= r\begin{bmatrix} X_{01}-X_{02} & L_{A_1} & L_{A_2} & A_1^{+}S_1 & A_2^{+}S_2 \\ R_{A_1^{\eta*}} & 0 & 0 & 0 & 0 \\ R_{A_2^{\eta*}} & 0 & 0 & 0 & 0 \\ S_1^{\eta*}\left(A_1^{+}\right)^{\eta*} & 0 & 0 & 0 & 0 \\ S_2^{\eta*}\left(A_2^{+}\right)^{\eta*} & 0 & 0 & 0 & 0 \end{bmatrix} \\
&= r\begin{bmatrix} X_{01}-X_{02} & I_n & I_n & A_1^{+}B_1L_{M_1} & A_2^{+}B_2L_{M_2} & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & A_1^{\eta*} & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & A_2^{\eta*} \\ R_{M_1^{\eta*}}B_1^{\eta*}\left(A_1^{+}\right)^{\eta*} & 0 & 0 & 0 & 0 & 0 & 0 \\ R_{M_2^{\eta*}}B_2^{\eta*}\left(A_2^{+}\right)^{\eta*} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_2 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&\quad - 2r\left(A_1\right) - 2r\left(A_2\right) \\
&= r\begin{bmatrix} 0 & I_n & I_n & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & 0 & A_1^{\eta*} & 0 \\ I_n & 0 & 0 & 0 & 0 & 0 & A_2^{\eta*} \\ 0 & 0 & 0 & 0 & 0 & -R_{M_1^{\eta*}}B_1^{\eta*} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -R_{M_2^{\eta*}}B_2^{\eta*} \\ 0 & A_1 & 0 & -B_1L_{M_1} & 0 & C_1 & 0 \\ 0 & 0 & A_2 & 0 & -B_2L_{M_2} & 0 & -C_2 \end{bmatrix} \\
&\quad - 2r\left(A_1\right) - 2r\left(A_2\right) \\
&= r\begin{bmatrix} 0 & 0 & 0 & -A_1^{\eta*} & A_2^{\eta*} \\ 0 & 0 & 0 & -R_{M_1^{\eta*}}B_1^{\eta*} & 0 \\ 0 & 0 & 0 & 0 & -R_{M_2^{\eta*}}B_2^{\eta*} \\ -A_1 & -B_1L_{M_1} & 0 & A_1X_{01}A_1^{\eta*} & 0 \\ A_2 & 0 & -B_2L_{M_2} & 0 & -A_2X_{02}A_2^{\eta*} \end{bmatrix} \\
&\quad - 2r\left(A_1\right) - 2r\left(A_2\right) + 2n \\
&= r\begin{bmatrix} 0 & 0 & 0 & -A_1^{\eta*} & A_2^{\eta*} & 0 & 0 \\ 0 & 0 & 0 & -B_1^{\eta*} & 0 & M_1^{\eta*} & 0 \\ 0 & 0 & 0 & 0 & -B_2^{\eta*} & 0 & M_2^{\eta*} \\ -A_1 & -B_1 & 0 & C_1 & 0 & 0 & 0 \\ A_2 & 0 & -B_2 & 0 & -C_2 & 0 & 0 \\ 0 & M_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & M_2 & 0 & 0 & 0 & 0 \end{bmatrix} \\
&\quad - 2r\left(A_1\right) - 2r\left(A_2\right) + 2n - 2r\left(M_1\right) - 2r\left(M_2\right)
\end{aligned}
$$

$$
= 2n + r \begin{bmatrix} 0 & 0 & 0 & -A_1^{\eta*} & A_2^{\eta*} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -B_1^{\eta*} & 0 & B_1^{\eta*} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -B_2^{\eta*} & 0 & B_2^{\eta*} & 0 & 0 \\ -A_1 & -B_1 & 0 & C_1 & 0 & 0 & 0 & 0 & 0 \\ A_2 & 0 & -B_2 & 0 & -C_2 & 0 & 0 & 0 & 0 \\ 0 & B_1 & 0 & 0 & 0 & 0 & 0 & A_1 & 0 \\ 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 & A_2 \\ 0 & 0 & 0 & 0 & 0 & A_1^{\eta*} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_2^{\eta*} & 0 & 0 \end{bmatrix}
$$
$$
- 2r \begin{bmatrix} B_1 & A_1 \end{bmatrix} - 2r \begin{bmatrix} B_2 & A_2 \end{bmatrix} - 2r(A_1) - 2r(A_2)
$$
$$
= r \begin{bmatrix} 0 & 0 & 0 & A_1^{\eta*} & -A_2^{\eta*} \\ 0 & 0 & 0 & B_1^{\eta*} & 0 \\ 0 & 0 & 0 & 0 & B_2^{\eta*} \\ A_1 & B_1 & 0 & C_1 & 0 \\ A_2 & 0 & B_2 & 0 & C_2 \end{bmatrix} + 2n - 2r \begin{bmatrix} B_1 & A_1 \end{bmatrix} - 2r \begin{bmatrix} B_2 & A_2 \end{bmatrix}
$$
$$
= 2n + r(L_1) - 2r \begin{bmatrix} B_1 & A_1 \end{bmatrix} - 2r \begin{bmatrix} B_2 & A_2 \end{bmatrix}. \tag{2.8}
$$

$$
r \begin{bmatrix} Y_{01} - Y_{02} & N_2 & P_2 \\ N_2^{\eta*} & 0 & 0 \end{bmatrix}
$$
$$
= r \begin{bmatrix} Y_{01} - Y_{02} & L_{B_1} & L_{B_2} & L_{M_1} L_{S_1} & L_{M_2} L_{S_2} & L_{M_1} & L_{M_2} \\ L_{B_1}^{\eta} & 0 & 0 & 0 & 0 & 0 & 0 \\ L_{B_2}^{\eta} & 0 & 0 & 0 & 0 & 0 & 0 \\ (L_{S_1})^{\eta} (L_{M_1})^{\eta} & 0 & 0 & 0 & 0 & 0 & 0 \\ (L_{S_2})^{\eta} (L_{M_2})^{\eta} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$
$$
= r \begin{bmatrix} Y_{01} - Y_{02} & L_{B_1} & L_{B_2} & L_{M_1} & L_{M_2} \\ R_{B_1^{\eta*}} & 0 & 0 & 0 & 0 \\ R_{B_2^{\eta*}} & 0 & 0 & 0 & 0 \\ R_{S_1^{\eta*}} (L_{M_1})^{\eta} & 0 & 0 & 0 & 0 \\ R_{S_2^{\eta*}} (L_{M_2})^{\eta} & 0 & 0 & 0 & 0 \end{bmatrix}
$$
$$
= r \begin{bmatrix} Y_{01} - Y_{02} & I_k & I_k & I_k & I_k & 0 & 0 & 0 & 0 \\ I_k & 0 & 0 & 0 & 0 & B_1^{\eta*} & 0 & 0 & 0 \\ I_k & 0 & 0 & 0 & 0 & 0 & B_2^{\eta*} & 0 & 0 \\ (L_{M_1})^{\eta} & 0 & 0 & 0 & 0 & 0 & 0 & S_1^{\eta*} & 0 \\ (L_{M_2})^{\eta} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & S_2^{\eta*} \\ 0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_2 & 0 & 0 & 0 & 0 \end{bmatrix}
$$
$$
- 2r(B_1) - 2r(B_2) - r(S_1) - r(S_2) - r(M_1) - r(M_2)
$$

$$= r\begin{bmatrix} 0 & I_k & I_k & I_k & I_k & 0 & 0 & 0 & 0 \\ I_k & 0 & 0 & 0 & 0 & B_1^{\eta*} & 0 & 0 & 0 \\ I_k & 0 & 0 & 0 & 0 & 0 & B_2^{\eta*} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\left(L_{M_1}\right)^{\eta} B_1^{\eta*} & 0 & S_1^{\eta*} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\left(L_{M_2}\right)^{\eta} B_2^{\eta*} & 0 & S_2^{\eta*} \\ -B_1Y_{01} & B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ B_2Y_{02} & 0 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_2 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$-2r\left(B_1\right)-2r\left(B_2\right)-r\left(S_1\right)-r\left(S_2\right)-r\begin{bmatrix} A_1 & B_1 \end{bmatrix}+r\left(A_1\right)-r\begin{bmatrix} A_2 & B_2 \end{bmatrix}$$
$$+r\left(A_2\right)$$
$$= r\begin{bmatrix} 0 & I_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -B_1^{\eta*} & B_2^{\eta*} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & S_1^{\eta*} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & S_2^{\eta*} \\ 0 & 0 & -B_1 & -B_1 & -B_1 & C_1 & 0 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 & 0 & -C_2 & 0 & 0 \\ 0 & 0 & 0 & R_{A_1}B_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & R_{A_2}B_2 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$-2r\left(B_1\right)-2r\left(B_2\right)-r\left(S_1\right)-r\left(S_2\right)-r\begin{bmatrix} A_1 & B_1 \end{bmatrix}+r\left(A_1\right)-r\begin{bmatrix} A_2 & B_2 \end{bmatrix}$$
$$+r\left(A_2\right)$$
$$= 2k + r\begin{bmatrix} 0 & 0 & 0 & -B_1^{\eta*} & B_2^{\eta*} \\ -B_1 & -B_1 & -B_1 & C_1 & 0 \\ B_2 & 0 & 0 & 0 & -C_2 \\ 0 & R_{A_1}B_1 & 0 & 0 & 0 \\ 0 & 0 & R_{A_2}B_2 & 0 & 0 \end{bmatrix}$$
$$-2r\left(B_1\right)-2r\left(B_2\right)-r\begin{bmatrix} A_1 & B_1 \end{bmatrix}+r\left(A_1\right)-r\begin{bmatrix} A_2 & B_2 \end{bmatrix}+r\left(A_2\right)$$
$$= 2k + r\begin{bmatrix} 0 & 0 & 0 & -B_1^{\eta*} & B_2^{\eta*} & 0 & 0 \\ -B_1 & -B_1 & -B_1 & C_1 & 0 & 0 & 0 \\ B_2 & 0 & 0 & 0 & -C_2 & 0 & 0 \\ 0 & B_1 & 0 & 0 & 0 & A_1 & 0 \\ 0 & 0 & B_2 & 0 & 0 & 0 & A_2 \end{bmatrix}$$
$$-2r\left(B_1\right)-2r\left(B_2\right)-r\begin{bmatrix} A_1 & B_1 \end{bmatrix}-r\begin{bmatrix} A_2 & B_2 \end{bmatrix}$$
$$= 2k + r\begin{bmatrix} B_1^{\eta*} & B_2^{\eta*} & 0 & 0 & 0 & 0 & 0 \\ C_1 & 0 & -B_1 & 0 & 0 & 0 & 0 \\ 0 & C_2 & B_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 & 0 & A_1 & 0 \\ 0 & 0 & 0 & 0 & B_2 & 0 & A_2 \end{bmatrix}$$
$$-2r\left(B_1\right)-2r\left(B_2\right)-r\begin{bmatrix} A_1 & B_1 \end{bmatrix}-r\begin{bmatrix} A_2 & B_2 \end{bmatrix}$$
$$= 2k + r\left(D_2\right)-2r\left(B_1\right)-2r\left(B_2\right)-r\begin{bmatrix} A_1 & B_1 \end{bmatrix}-r\begin{bmatrix} A_2 & B_2 \end{bmatrix}. \qquad (2.9)$$

$$r\begin{bmatrix} Y_{01}-Y_{02} & N_2 & P_2 \\ N_2^{\eta} & 0 & 0 \\ P_2^{\eta*} & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} Y_{01}-Y_{02} & L_{B_1} & L_{B_2} & L_{M_1}L_{S_1} & L_{M_2}L_{S_2} & L_{M_1} & L_{M_2} \\ L_{B_1}^{\eta} & 0 & 0 & 0 & 0 & 0 & 0 \\ L_{B_2}^{\eta} & 0 & 0 & 0 & 0 & 0 & 0 \\ (L_{S_1})^{\eta}(L_{M_1})^{\eta} & 0 & 0 & 0 & 0 & 0 & 0 \\ (L_{S_2})^{\eta}(L_{M_2})^{\eta} & 0 & 0 & 0 & 0 & 0 & 0 \\ (L_{M_1})^{\eta} & 0 & 0 & 0 & 0 & 0 & 0 \\ (L_{M_2})^{\eta} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} Y_{01}-Y_{02} & L_{B_1} & L_{B_2} & L_{M_1} & L_{M_2} \\ R_{B_1^{\eta*}} & 0 & 0 & 0 & 0 \\ R_{B_2^{\eta*}} & 0 & 0 & 0 & 0 \\ R_{M_1^{\eta*}} & 0 & 0 & 0 & 0 \\ R_{M_2^{\eta*}} & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= r\begin{bmatrix} Y_{01}-Y_{02} & I_k & I_k & I_k & I_K & 0 & 0 & 0 & 0 \\ I_k & 0 & 0 & 0 & 0 & B_1^{\eta*} & 0 & 0 & 0 \\ I_k & 0 & 0 & 0 & 0 & 0 & B_2^{\eta*} & 0 & 0 \\ I_k & 0 & 0 & 0 & 0 & 0 & 0 & M_1^{\eta*} & 0 \\ I_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_2^{\eta*} \\ 0 & B_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-2r(B_1)-2r(B_2)-2r(M_1)-2r(M_2)$$

$$= r\begin{bmatrix} 0 & I_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_k & 0 & 0 & 0 & 0 & B_1^{\eta*} & 0 & 0 & 0 \\ I_k & 0 & 0 & 0 & 0 & 0 & B_2^{\eta*} & 0 & 0 \\ I_k & 0 & 0 & 0 & 0 & 0 & 0 & M_1^{\eta*} & 0 \\ I_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & M_2^{\eta*} \\ -B_1Y_{01} & B_1 & -B_1 & -B_1 & -B_1 & 0 & 0 & 0 & 0 \\ B_2Y_{02} & 0 & B_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & M_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & M_2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$-2r(B_1)-2r(B_2)-2r(M_1)-2r(M_2)$$

$$= r\begin{bmatrix} 0 & 0 & 0 & -B_1^{\eta*} & B_2^{\eta*} & 0 & 0 \\ 0 & 0 & 0 & -B_1^{\eta*} & 0 & B_1^{\eta*}L_{A_1^{\eta*}} & 0 \\ 0 & 0 & 0 & -B_1^{\eta*} & 0 & 0 & B_2^{\eta*}L_{A_2^{\eta*}} \\ -B_1 & -B_1 & -B_1 & C_1 & 0 & 0 & 0 \\ B_2 & 0 & 0 & 0 & -C_2 & 0 & 0 \\ 0 & R_{A_1}B_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & R_{A_2}B_2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$
\begin{aligned}
&2k-2r\left(B_1\right)-2r\left(B_2\right)-2r\left(M_1\right)-2r\left(M_2\right)\\
&=r\left[\begin{array}{ccccccccc}
0 & 0 & 0 & -B_1^{\eta*} & B_2^{\eta*} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -B_1^{\eta*} & 0 & B_1^{\eta*} & 0 & 0 & 0\\
0 & 0 & 0 & -B_1^{\eta*} & 0 & 0 & B_2^{\eta*} & 0 & 0\\
-B_1 & -B_1 & -B_1 & C_1 & 0 & 0 & 0 & 0 & 0\\
B_2 & 0 & 0 & 0 & -C_2 & 0 & 0 & 0 & 0\\
0 & B_1 & 0 & 0 & 0 & 0 & 0 & A_1 & 0\\
0 & 0 & B_2 & 0 & 0 & 0 & 0 & 0 & A_2\\
0 & 0 & 0 & 0 & 0 & A_1^{\eta*} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & A_2^{\eta*} & 0 & 0
\end{array}\right]\\
&\quad-2r\left(B_1\right)-2r\left(B_2\right)-2r\left(M_1\right)-2r\left(M_2\right)-2r\left(A_1\right)-2r\left(A_2\right)\\
&=r\left[\begin{array}{ccccccccc}
0 & I_k & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
I_k & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -B_1^{\eta*} & B_2^{\eta*} & 0 & 0\\
0 & 0 & 0 & 0 & 0 & -B_1^{\eta*} & 0 & M_1^{\eta*} & 0\\
0 & 0 & 0 & 0 & 0 & -B_1^{\eta*} & 0 & 0 & M_2^{\eta*}\\
0 & 0 & -B_1 & -B_1 & -B_1 & B_1Y_{01}B_1^{\eta*} & 0 & 0 & 0\\
0 & 0 & B_2 & 0 & 0 & 0 & -B_2Y_{02}B_2^{\eta*} & 0 & 0\\
0 & 0 & 0 & M_1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & M_2 & 0 & 0 & 0 & 0
\end{array}\right]\\
&\quad-2r\left(B_1\right)-2r\left(B_2\right)-2r\left(M_1\right)-2r\left(M_2\right)\\
&=2k+r\left[\begin{array}{ccccccccc}
0 & 0 & 0 & 0 & B_2^{\eta*} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & B_1^{\eta*} & 0 & 0 & B_2^{\eta*} & 0 & 0\\
0 & 0 & -B_1 & C_1 & 0 & 0 & 0 & 0 & 0\\
B_2 & 0 & 0 & 0 & C_2 & 0 & 0 & 0 & 0\\
0 & B_1 & 0 & 0 & 0 & 0 & 0 & A_1 & 0\\
0 & 0 & B_2 & 0 & 0 & A_1^{\eta*} & 0 & 0 & A_2\\
0 & 0 & 0 & 0 & 0 & 0 & A_2^{\eta*} & 0 & 0
\end{array}\right]\\
&\quad-2r\left(B_1\right)-2r\left(B_2\right)-2r\left[\begin{array}{cc} A_1 & B_1\end{array}\right]-2r\left[\begin{array}{cc} A_2 & B_2\end{array}\right]\\
&=2k+r\left(L_2\right)-2r\left(B_1\right)-2r\left(B_2\right)-2r\left[\begin{array}{cc} A_1 & B_1\end{array}\right]-2r\left[\begin{array}{cc} A_2 & B_2\end{array}\right]. \qquad (2.10)
\end{aligned}
$$

$$
\begin{aligned}
r\left(N_1\right)&=r\left[\begin{array}{cc} L_{A_1} & L_{A_2}\end{array}\right]\\
&=r\left[\begin{array}{cc} I_n & I_n\\ A_1 & 0\\ 0 & A_2\end{array}\right]-r\left(A_1\right)-r\left(A_2\right)\\
&=r\left[\begin{array}{c} A_1\\ A_2\end{array}\right]-r\left(A_1\right)-r\left(A_2\right)+n. \qquad (2.11)
\end{aligned}
$$

$$
r\left(N_2\right)=r\left[\begin{array}{cccc} L_{B_1} & L_{B_2} & L_{M_1}L_{S_1} & L_{M_2}L_{S_2}\end{array}\right]
$$

$$= r\begin{bmatrix} I_k & I_k & L_{M_1} & L_{M_2} \\ B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & S_1 & 0 \\ 0 & 0 & 0 & S_2 \end{bmatrix} - r(B_1) - r(B_2) - r(S_1) - r(S_2)$$

$$= r\begin{bmatrix} I_k & I_k & 0 & 0 \\ B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & S_1 & 0 \\ 0 & 0 & 0 & S_2 \end{bmatrix} - r(B_1) - r(B_2) - r(S_1) - r(S_2)$$

$$= r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - r(B_1) - r(B_2) + k. \tag{2.12}$$

Substituting (2.7), (2.8), (2.11) and (2.9), (2.10), (2.12) into (2.5) and (2.6), respectively, we get (2.1) and (2.2). □

Corollary 2.2. *Let $A_i \in \mathbb{H}^{m_i \times n}$, $B_i \in \mathbb{H}^{m_i \times k}$, and $C_i = C_i^{\eta *} \in \mathbb{H}^{m_i \times m_i}$ $(i = 1, 2)$ be given, and let D_1, D_2, L_1, and L_2 be as given in Theorem 2.1. Assume that the pair of quaternion matrix equations in (1.3) has an η-Hermitian solution. Then the following properties hold:*
a) The system of quaternion matrix equations (1.3) has a common η-Hermitian solution for X if and only if

$$2r(D_1) = r(L_1) + 2r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.$$

b) The system of quaternion matrix equations (1.3) has a common η-Hermitian solution for Y if and only if

$$2r(D_2) = r(L_2) + 2r\begin{bmatrix} B_1 \\ B_2 \end{bmatrix}.$$

By vanishing some matrices in (1.3), we obtain necessary and sufficient conditions of the system (1.4) to have common η-Hermitian solution.

Corollary 2.3. *Let $A_i \in \mathbb{H}^{m_i \times n}$ and $C_i = C_i^{\eta *} \in \mathbb{H}^{m_i \times m_i}$ $(i = 1,\ 2)$ be given. Assume that both of matrix equations in (1.4) is consistent. Then, the system (1.4) has a common η-Hermitian solution if and only if*

$$r\begin{bmatrix} 0 & A_1^{\eta *} & A_2^{\eta *} \\ A_1 & C_1 & 0 \\ A_2 & 0 & -C_2 \end{bmatrix} = 2r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}.$$

3. Extremal ranks of the matrix expression $C_2 - A_2 X A_2^{\eta *}$ with respect to η-Hermitian solution to (1.1)

In this section, we derive the extremal ranks of the η-Hermitian matrix expression

$$f(X) = C_2 - A_2 X_1 A_2^{\eta *} \tag{3.1}$$

subject to η-Hermitian solution of quaternion matrix equation (1.1), where $A_i \in \mathbb{H}^{m_i\times n}$ and $C_i = C_i^{\eta*} \in \mathbb{H}^{m_i\times m_i}$ for $(i = 1, 2)$.

Theorem 3.1. *Let $f(X)$ be as given in (3.1). The extermal ranks of the quaternion matrix expression $f(X)$ subject to the consistent equation (1.1) are as follows:*

$$\max_{A_1X_1A_1^{\eta*}=C_1} r\left(f\left(X\right)\right) = \min\left\{ r\begin{bmatrix} C_2 & A_2 \end{bmatrix}, r\begin{bmatrix} C_2 & A_2 & 0 \\ A_2^{\eta*} & 0 & A_1^{\eta*} \\ 0 & A_1 & -C_1 \end{bmatrix} - 2r\left(A_1\right) \right\}, \tag{3.2}$$

$$\min_{A_1X_1A_1^{\eta*}=C_1} r\left(f\left(X\right)\right) = 2r\begin{bmatrix} C_2 & A_2 \end{bmatrix} + r\begin{bmatrix} C_2 & A_2 & 0 \\ A_2^{\eta*} & 0 & A_1^{\eta*} \\ 0 & A_1 & -C_1 \end{bmatrix} - 2r\begin{bmatrix} C_2 & A_2 \\ A_2^{\eta*} & 0 \\ 0 & A_1 \end{bmatrix}. \tag{3.3}$$

Proof. By Lemma 1.4, the quaternion matrix equation (1.1) has an η-Hermitian solution if and only if $A_1A_1^{+}C_1 = C_1$. In this case, the η-Hermitian solution can be expressed as

$$X_1 = A_1^{+}C_1\left(A_1^{+}\right)^{\eta*} + L_{A_1}U + U^{\eta*}\left(L_{A_1}\right)^{\eta*}, \tag{3.4}$$

where U is an arbitrary matrix over $\mathbb{H}$ with appropriate size.
Substituting (3.4) into (3.1) yields

$$\begin{aligned} f\left(X\right) &= C_2 - A_2A_1^{+}C_1\left(A_1^{+}\right)^{\eta*}A_2^{\eta*} - A_2L_{A_1}UA_2^{\eta*} - \left(A_2L_{A_1}UA_2^{\eta*}\right)^{\eta*} \\ &= G - SUA_2^{\eta*} - \left(SUA_2^{\eta*}\right)^{\eta*}, \end{aligned}$$

where $G = C_2 - A_2A_1^{+}C_1\left(A_1^{+}\right)^{\eta*}A_2^{\eta*}$, $S = A_2L_{A_1}$.
It follows from Lemma 1.6 that

$$\begin{aligned} \max_{A_1X_1A_1^{\eta*}=C_1} r\ f\left(X\right) &= \max_{U} r\left[G - SUA_2^{\eta*} - \left(SUA_2^{\eta*}\right)^{\eta*}\right] \\ &= \min\left\{ r\begin{bmatrix} G & A_2 \end{bmatrix}, r\begin{bmatrix} G & S \\ S^{\eta*} & 0 \end{bmatrix} \right\}, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \min_{A_1X_1A_1^{\eta*}=C_1} r\ f\left(X\right) &= \min_{U} r\begin{bmatrix} C_2 - A_2A_1^{+}C_1\left(A_1^{+}\right)^{\eta*}A_2^{\eta*} - A_2L_{A_1}UA_2^{\eta*} \\ -\left(A_2L_{A_1}UA_2^{\eta*}\right)^{\eta*} \end{bmatrix} \\ &= 2r\begin{bmatrix} G & A_2 \end{bmatrix} + r\begin{bmatrix} G & S \\ S^{\eta*} & 0 \end{bmatrix} - 2r\begin{bmatrix} G & S \\ A_2^{\eta*} & 0 \end{bmatrix}. \end{aligned} \tag{3.6}$$

Applying Lemma 1.1, block Gaussian eliminations and simplifying by $A_1A_1^{+}C_1 = C_1$, we obtain

$$r\begin{bmatrix} G & A_2 \end{bmatrix} = r\begin{bmatrix} C_2 - A_2A_1^{+}C_1\left(A_1^{+}\right)^{\eta*}A_2^{\eta*} & A_2 \end{bmatrix} = r\begin{bmatrix} C_2 & A_2 \end{bmatrix} \tag{3.7}$$

$$r\begin{bmatrix} C_2 - A_2A_1^{+}C_1\left(A_1^{+}\right)^{\eta*}A_2^{\eta*} & A_2L_{A_1} \\ A_2^{\eta*} & 0 \end{bmatrix} = r\begin{bmatrix} C_2 & A_2 \\ A_2^{\eta*} & 0 \\ 0 & A_1 \end{bmatrix} - r\left(A_1\right). \tag{3.8}$$

$$r\begin{bmatrix} C_2 - A_2A_1^{+}C_1\left(A_1^{+}\right)^{\eta*}A_2^{\eta*} & A_2L_{A_1} \\ \left(A_2L_{A_1}\right)^{\eta*} & 0 \end{bmatrix}$$

$$= r \begin{bmatrix} C_2 - A_2 A_1^{+} C_1 \left(A_1^{+}\right)^{\eta*} A_2^{\eta*} & A_2 L_{A_1} \\ R_{A_1^{\eta*}} A_2^{\eta*} & 0 \end{bmatrix}$$

$$= r \begin{bmatrix} C_2 - A_2 A_1^{+} C_1 \left(A_1^{+}\right)^{\eta*} A_2^{\eta*} & A_2 & 0 \\ A_2^{\eta*} & 0 & A_1^{\eta*} \\ 0 & A_1 & 0 \end{bmatrix} - 2r\left(A_1\right)$$

$$= r \begin{bmatrix} C_2 & A_2 & 0 \\ A_2^{\eta*} & 0 & A_1^{\eta*} \\ 0 & A_1 & -C_1 \end{bmatrix} - 2r\left(A_1\right). \tag{3.9}$$

Substituting (3.7)–(3.9) into (3.5) and (3.6) yields the desired results in (3.2) and (3.3). □

In the previous theorem, if the quaternion matrix equation $A_2 X_2 A_2^{\eta*} = C_2$ is consistent, that is, $A_2 A_2^{+} C_2 = C_2$, then we have the following results.

Corollary 3.2. *Assume that both quaternion matrix equations $A_1 X_1 A_1^{\eta*} = C_1$ and $A_2 X_2 A_2^{\eta*} = C_2$ are consistent. Then*

$$\max_{A_1 X_1 A_1^{\eta*} = C_1} r\left(C_2 - A_2 X_1 A_2^{\eta*}\right) = \min\left\{ r\left(A_2\right), r \begin{bmatrix} C_2 & A_2 & 0 \\ A_2^{\eta*} & 0 & A_1^{\eta*} \\ 0 & A_1 & -C_1 \end{bmatrix} - 2r\left(A_1\right) \right\}, \tag{3.10}$$

$$\min_{A_1 X_1 A_1^{\eta*} = C_1} r\left(C_2 - A_2 X_1 A_2^{\eta*}\right) = r \begin{bmatrix} C_2 & A_2 & 0 \\ A_2^{\eta*} & 0 & A_1^{\eta*} \\ 0 & A_1 & -C_1 \end{bmatrix} - 2r \begin{bmatrix} A_2 \\ A_1 \end{bmatrix}. \tag{3.11}$$

Corollary 3.3. *Let the rank equality in (3.11) equal zero. Then we obtain the same result of Corollary 2.3.*

As is well known, for a given block matrix

$$M = \begin{bmatrix} A & B \\ B^{\eta*} & D \end{bmatrix},$$

where A and D are η-Hermitian quaternion matrices with appropriate sizes, the Hermitian Schur complement of A in M is defined as

$$S_A = D - B^{\eta*} A^{-} B, \tag{3.12}$$

where A^{-} is an η-Hermitian generalized inverse of A, that is,

$$A^{-} \in \left\{ X \mid AXA = A,\ X = X^{\eta*} \right\}.$$

Now, we use Theorem 3.1 to establish the extremal ranks of S_A given by (3.12) with respect to A_1^{-}, which is an η-Hermitian solution to the quaternion matrix equation (1.1).

Theorem 3.4. *Let $A_1 = A_1^{\eta*}, C_1 = C_1^{\eta*} \in \mathbb{H}^{n\times n}$, $B \in \mathbb{H}^{n\times m}$, and $D = D^{\eta*} \in \mathbb{H}^{m\times m}$ be given. Assume that quaternion matrix equation in (1.1) is consistent. Then*

$$\max_{A_1 A_1^{-} A_1^{\eta*} = C_1} r\left(S_A\right) = \min\left\{ r\begin{bmatrix} D & B^{\eta*} \end{bmatrix}, r\begin{bmatrix} D & B^{\eta*} \\ B & A_1 \end{bmatrix} - r\left(A_1\right) \right\}, \tag{3.13}$$

$$\min_{A_1 A_1^- A_1^{\eta*}=C_1} r\left(S_A\right)=2r\begin{bmatrix} D & B^{\eta*} \end{bmatrix}+r\begin{bmatrix} D & B^{\eta*} \\ B & A_1 \end{bmatrix}-2r\begin{bmatrix} D & B^{\eta*} \\ B & 0 \\ 0 & A_1 \end{bmatrix}+r\left(A_1\right). \tag{3.14}$$

Proof. It is obvious that

$$\max_{A_1 A_1^- A_1^{\eta*}=C_1} r\left(D-B^{\eta*}A_1^- B\right)=\max_{\substack{A_1 X A_1^{\eta*}=C_1\\ A_1 X A_1=A_1}} r\left(D-B^{\eta*}XB\right),$$

$$\min_{A_1 A_1^- A_1^{\eta*}=C_1} r\left(D-B^{\eta*}A_1^- B\right)=\min_{\substack{A_1 X A_1^{\eta*}=C_1\\ A_1 X A_1=A_1}} r\left(D-B^{\eta*}XB\right).$$

Thus, in Theorem 3.1, we set $A_2=B^{\eta*}$, $C_2=D$ and $A_1=A_1^{\eta*}=C_1$. Therefore, we get

$$\max_{A_1 A_1^- A_1^{\eta*}=C_1} r\left(D-B^{\eta*}A_1^- B\right)=\min\left\{\begin{array}{c} r\begin{bmatrix} D & B^{\eta*} \end{bmatrix}, r\begin{bmatrix} D & B^{\eta*} & 0 \\ B & 0 & A_1 \\ 0 & A_1 & -A_1 \end{bmatrix} \\ -2r\left(A_1\right) \end{array}\right\}, \tag{3.15}$$

$$\begin{aligned}\min_{A_1 A_1^- A_1^{\eta*}=C_1} r\left(D-B^{\eta*}A_1^- B\right)&=2r\begin{bmatrix} D & B^{\eta*} \end{bmatrix}+r\begin{bmatrix} D & B^{\eta*} & 0 \\ B & 0 & A_1 \\ 0 & A_1 & -A_1 \end{bmatrix}\\ &\quad-2r\begin{bmatrix} D & B^{\eta*} \\ B & 0 \\ 0 & A_1 \end{bmatrix}.\end{aligned} \tag{3.16}$$

Simplifying by Gaussian elimination, we have

$$r\begin{bmatrix} D & B^{\eta*} & 0 \\ B & 0 & A_1 \\ 0 & A_1 & -A_1 \end{bmatrix}=r\begin{bmatrix} D & B^{\eta*} & 0 \\ B & A_1 & 0 \\ 0 & 0 & -A_1 \end{bmatrix}=r\begin{bmatrix} D & B^{\eta*} \\ B & A_1 \end{bmatrix}+r\left(A_1\right). \tag{3.17}$$

Substituting (3.17) into (3.15) and (3.16), the proof is finished. □

Acknowledgement. The authors are grateful for the detailed comments from the referees, which significantly improved the quality of the paper.

References

[1] A. Ben Israel and T.N.E. Greville, *Generalized Inverse Theory and Applications*, Krieger, 1980.

[2] Z.H. He and Q.W. Wang, *A real quaternion matrix equation with applications*, Linear Multilinear Algebra, **61** (2013), no. 6, 725–740.

[3] I.A. Khan, Q.W. Wang, and G.-J. Song, *Minimal ranks of some quaternion matrix expressions with applications*, Appl. Math. Comput. **217** (2010), no. 5, 2031–2040.

[4] X. Liu and Z.H. He, *η-Hermitian solution to a system of quaternion matrix equations.* Bull. Malays. Math. Sci. Soc. **43** (2020), no. 6, 4007–4027.

[5] Y.H. Liu, *Ranks of solutions of the linear matrix equation $AX+YB=C$*. Comput. Math. Appl. **52** (2006), no. 6-7, 861–872.

[6] Y. Liu and Y. Tian, *Max-min problems on the ranks and inertias of the matrix expressions* $A-BXC\pm(BXC)^*$ *with applications.* J. Optim. Theory Appl. **148** (2011), no. 3, 593–622.

[7] G. Marsaglia and G.P.H. Styan, *Equalities and inequalities for ranks of matrices.* Linear and Multilinear Algebra, **2** (1974/75), 269–292.

[8] C.C. Took, D.P. Mandic and F.Z. Zhang, *On the unitary diagonalization of a special class of quaternion matrices*, Appl. Math. Lett. **24** (2011), 1806–1809.

[9] Q.W. Wang and C.K. Li, *Ranks and the least-norm of the general solution to a system of quaternion matrix equations*, Linear Algebra Appl. **430** (2009), no. 5-6, 1626–1640.

[10] S.W. Yu, *Ranks of a constrained Hermitian matrix expression with applications.* J. Appl. Math. **2013**, Art. ID 514984, 9 pp.

[11] X. Zhang, *The* η*-Hermitian solutions to some systems of real quaternion matrix equations.* Filomat, **36** (2022), no. 1, 315–330.

[1] Department of Mathematics and Informatics, Dynamic Systems and Control Laboratory, Faculty of Exact Sciences and Sciences of Nature and Life, University of Oum El Bouaghi, 04000, Algeria.

Email address: `radja.belkhiri@univ-oeb.dz`

[2] Department of Mathematics and Informatics, Dynamic Systems and Control Laboratory, Faculty of Exact Sciences and Sciences of Nature and Life, University of Oum El Bouaghi, 04000, Algeria.

Email address: `guerarra.siham@univ-oeb.dz`