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RATIONAL HOMOTOPY OF A MAP OF PROJECTIVE QUATERNIONS AND THEIR RELATIVE GOTTLIEB GROUPS

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Abstract. In this paper, we show in terms of Sullivan models that the rational homotopy of a map $\iota : \mathbb{H}P^m \hookrightarrow \mathbb{H}P^{m+r}$ between projective quaternion spaces is a product of a quaternion projective space and odd spheres. We also study the properties of a map $\operatorname{aut}_1 \mathbb{H}P^m \to \operatorname{maps}(\mathbb{H}P^m, \mathbb{H}P^{m+r}; \iota)$ and its G-sequence.

1. Introduction

Let $h : M \to N$ be a based map, where M and N are simply connected finite CW-complexes. As in [13], define by $\omega : \operatorname{maps}(M, N; h) \to N$ the *evaluation map*, where $\operatorname{maps}(M, N; h)$ is the component of h in the space of mappings from M to N, and by

$$\omega_\sharp : \pi_* \operatorname{maps}(M, N; h) \to \pi_*(N)$$

the image of the homomorphism induced in homotopy groups called the mth *evaluation subgroup of* h, and it is denoted by $G_m(N, M; h)$. In particular, if $h = id_M$, then the space $\operatorname{maps}(M, N; h)$ is the monoid $\operatorname{aut}_1(M)$ of self-equivalences of M homotopic to the identity of M, such that $ev : \operatorname{aut}_1(M) \to M$ is the evaluation map, and

$$ev_\sharp : \pi_*(\operatorname{aut}_1(M)) \to \pi_*(M)$$

is the image of the induced homomorphism called the mth Gottlieb group, denoted by $G_m(M)$ [9].

Furthermore, as it is known that a topological pair gives rise to a natural long exact sequence of homotopy groups, which plays an important role in relating homotopy groups of different topological spaces, but what about the subgroups

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of homotopy groups, that is, Gottlieb groups, generalized evaluation subgroups and relative evaluation subgroups? Thus, Woo and Lee [23] studied the properties of relative evaluation subgroups of a pair $G_m^{rel}(M, N; h)$ and proved that they fit in a sequence

$$\cdots \to G_{m+1}^{rel}(M, N; h) \to G_m(M) \to G_m(M, N; h) \to \cdots$$

called the G-sequence of h. This sequence is exact in some cases, for instance, if h is a homotopy monomorphism. Therefore, the exactness of the G-sequence relates subgroups of homotopy groups.

An important problem is then to describe the homotopy type of the mapping space $\operatorname{maps}(M, N; h)$ in terms of the homotopy types of M and N. In [10, 18], the authors described the rational homotopy classification problem for the components of some mapping spaces $\operatorname{maps}(M, N; h)$. In particular, Møller and Raussen [18] gave a different proof of our main result Theorem 1.1. On the other hand, most recently, in [13], the authors interpreted the homomorphism $\pi_* \operatorname{maps}(M, N; h) \to \pi_*(N)$ in terms of a map of chain complexes of derivations constructed directly from the Sullivan minimal model of h. As a result, following [13], the authors in [19, 14, 15, 7, 16, 24] used a map of chain complexes of derivations of minimal Sullivan models of mapping spaces to compute rational relative Gottlieb groups of some complex (resp., quaternion) Grassmannians. It is also known that these chain complexes are L_∞ models of mapping spaces (see [1, 2]). However, there are few explicit computations and descriptions known about rational Gottlieb groups of mapping spaces and their resulting G-sequence.

Thus, following [13, 2], the authors [8] studied the rational homotopy of function spaces between complex Grassmannians, whereas in [6], the main result provided another proof of the result in [18, Example 3.4] using the L_∞ model of a map $\iota : \mathbb{C}P^m \hookrightarrow \mathbb{C}P^{m+r}$.

In this note, our main result gives another proof of a result in [18, Example 3.4] using L_∞ models of mapping spaces. In the process, we also describe the associated G-sequence and the rational Gottlieb group of F_0-spaces that are rational two stage spaces. Hence, our main result reads as follows.

Theorem 1.1. *The mapping space* $\operatorname{maps}(\mathbb{H}P^m, \mathbb{H}P^{m+r}; \iota)$ *has the rational homotopy type of* $\mathbb{H}P^r \times S^{4r+7} \times \cdots \times S^{4(m+r)+3}$.

Considering the evaluation subgroups of the mapping $\operatorname{aut}_1 \mathbb{H}P^m \to \mathbb{H}P^{m+r}$, we have the following result.

Theorem 1.2. *The G-sequence of a map*

$$\operatorname{aut}_1 \mathbb{H}P^m \to \operatorname{maps}(\mathbb{H}P^m, \mathbb{H}P^{m+r}; \iota)$$

is not exact.

2. Preliminaries

Throughout this paper, our study is based on minimal Sullivan models in rational homotopy theory for which [3] is the main reference. All vector spaces and algebras are taken over a field of rational numbers $\mathbb{Q}$. We begin by reminding some standard definitions.

Definition 2.1. A commutative graded differential algebra (cdga) is a graded algebra (C, d) such that $ab = (-1)^{|a||b|}ba$ and $d(ab) = (da)b + (-1)^{|pq|}a(db)$ for all $a \in C^p, b \in C^q$. It is connected if $H^0(C) \cong \mathbb{Q}$. If $W = \oplus_{i\geq 1} W^i$ with $W^{\text{even}} := \oplus_{i\geq 1} W^{2i}$ and $W^{\text{odd}} := \oplus_{i\geq 1} W^{2i-1}$, then $\wedge W$ denotes the free commutative graded algebra defined by the tensor product

$$\wedge W = S(W^{\text{even}}) \otimes E(W^{\text{odd}}),$$

where $S(W^{\text{even}})$ is the symmetric algebra on W^{even} and $E(W^{\text{odd}})$ is the exterior algebra on W^{odd}.

Definition 2.2. A commutative differential graded algebra $(\wedge W, d)$ is a Sullivan algebra whenever $W = \cup_{k\geq 0} W(k)$ and $W(0) \subset W(1) \cdots$ such that $dW(0) = 0$ and $dW(k) \subset \wedge W(k-1)$. It is called minimal if $dW \subset \wedge^{\geq 2} W$.

If M is a simply connected space, then there is a cdga $A_{PL}(M)$ of rational polynomial differential forms on M that uniquely determines the rational homotopy type of M [22, 3].

Let (C, d) be a cdga. A derivation θ of degree r is a linear mapping $\theta : C^m \to C^{m-r}$ such that $\theta(xy) = \theta(x)y + (-1)^{r|x|}x\theta(y)$. Denote by $\operatorname{Der}_r C$ the vector space of all derivations of degree r, and $\operatorname{Der} C = \oplus_r \operatorname{Der}_k C$. The differential δ is defined in the usual way by $\partial\theta = d \circ \theta + (-1)^{r+1}\theta \circ d$. Let $(\wedge V, d)$ be a Sullivan algebra, where V is spanned by $\{v_1 \ldots, v_k\}$. Then, $\operatorname{Der} \wedge V$ is spanned by $\theta_1, \ldots, \theta_k$, where θ_i is the unique derivation of $\wedge V$ defined by $\theta_i(v_j) = \delta_{ij}$. The derivation θ_i will be denoted by $(v_i, 1)$. Moreover, an element $v \in V \cong \pi_*(X) \otimes \mathbb{Q}$ is a Gottlieb element of $\pi_*(X) \otimes \mathbb{Q}$ if and only if there is a derivation θ of $\wedge V$ satisfying $\theta(v) = 1$ and such that $\delta\theta = 0$ [3, p. 392].

Let $f : (C, d) \to (E, d)$ be a morphism of cdgas. An f-derivation of degree r is a linear mapping $\theta : C^m \to E^{m-r}$ for which $\theta(xy) = \theta(x)f(y) + (-1)^{r|x|}f(x)\theta(y)$. Denote by $\operatorname{Der}(C, E; f) = \oplus_m \operatorname{Der}_m(C, E; f)$ the graded vector space of all f-derivations. The differential graded vector space of all positive f-derivations is denoted by $(\operatorname{Der}(C, E; f), \partial)$, and the differential ∂ is defined by $\delta\theta = d_E \circ \theta + (-1)^{k+1}\theta \circ d_C$, where in degree one, we restrict to the subspace of cycles in $\operatorname{Der}_1(C, E; f)$.

It was shown in [13] that a pre-composition with f gives a chain complex map $f^* : \operatorname{Der}(E, E; 1) \to \operatorname{Der}(C, E; f)$ and that a post-composition with the augmentation $\varepsilon : E \to \mathbb{Q}$ gives a chain complex map $\varepsilon_* : \operatorname{Der}(C, E; f) \to \operatorname{Der}(C, \mathbb{Q}; \varepsilon)$. The evaluation subgroup of f is defined as follows:

$$G_m(C, E; f) = \operatorname{Im}\{H(\varepsilon_*) : H_m(\operatorname{Der}(C, E; f)) \to H_m(\operatorname{Der}(C, \mathbb{Q}; \varepsilon))\}.$$

In the case when $C = E$ and $f = 1_E$, we get the Gottlieb group of (E, d) defined as

$$G_m(E) = \operatorname{Im}\{H(\varepsilon_*) : H_m(\operatorname{Der}(E, E; 1)) \to H_m(\operatorname{Der}(E, \mathbb{Q}; \varepsilon))\}.$$

In particular, $G_m(E) \cong G_m(M_{\mathbb{Q}})$ if E is the minimal Sullivan model of a simply connected space M [3, Proposition 29.8].

Definition 2.3. A simply connected space M is called formal (see [4]) if there is a quasi-isomorphism $(\wedge W, d) \to H^*(\wedge W, d)$, where $(\wedge W, d)$ is the minimal Sullivan model of M.

Examples of formal spaces include spheres, quaternion projective spaces, homogeneous spaces G/H, where G and H have equal rank, and compact Kähler manifolds. Moreover, a product of formal spaces is also formal.

Definition 2.4. A finite simply connected CW-complex M of which the rational homotopy group $\pi_*(M)\otimes\mathbb{Q}$ is finite dimensional and the rational cohomology is evenly graded is called an F_0-space (see [20]).

Examples of F_0-spaces include finite products of even dimensional spheres, finite products of complex (resp., quaternion) projective spaces, homogeneous spaces G/H, where H is a closed subgroup of maximal rank of a compact connected Lie group G.

In [11, 20], the minimal Sullivan model of an F_0-space M is of the form $(\wedge V, d) = (\wedge(V_0 \oplus V_1), d)$, where V is finite dimensional and $dV_0 = 0$, $dV_1 \subseteq \wedge V_0$. Denote by $< v_1, v_2, \dots, v_n >$ the vector space generated by a finite basis $\{v_i\}$ of V. Write $V_0^{\text{even}} = \mathbb{Q} < x_1, \dots, x_n >= P$ and $V_1^{\text{odd}} = \mathbb{Q} < y_1, \dots, y_n >= W$, so that $(\wedge(V_0 \oplus V_1), d) \overset{\cong}{\to} (\wedge(P \oplus W), d)$ and $dP = 0$, $dW \subseteq \wedge P$. The associated minimal Sullivan model for an F_0-space M is a *two-stage* model. Moreover,

$$H^*(\wedge V, d) = \frac{\wedge(x_1, \dots, x_n)}{(\alpha_1, \dots, \alpha_n)},$$

where $(\alpha_1, \dots, \alpha_n)$ is a regular sequence in $\wedge P$. Hence, M admits a minimal Sullivan model of the form $(\wedge V, d) = (\wedge(P \oplus W), d)$, where $dP = 0$ and $dy_n = \alpha_n$. Thus, F_0-spaces are formal.

3. L_∞-models of mapping spaces

Here we recall some standard definitions on L_∞ algebras were introduced by Lada and Markl [12] and L_∞ models of function spaces studied by Buijs, Félix, and Murillo [1, 2].

Definition 3.1. A *permutation* $\sigma \in S_r$ is an $(m, k-m)$ shuffle if $\sigma(1) < \cdots < \sigma(m)$ and $\sigma(m+1) < \cdots < \sigma(r)$, where $m = 1, \dots, i$. The *Koszul sign* $\epsilon(\sigma)$ is determined by

$$y_1 \wedge \cdots \wedge y_k = \epsilon(\sigma) y_{\sigma(1)} \wedge \cdots \wedge y_{\sigma(k)},$$

where the subscripts indicate the degrees of the graded objects $y_1, \dots, y_r$.

Definition 3.2. [1] An L_∞ *algebra* is a graded vector space $L = \oplus_i L_i$ equipped with a family of linear maps

$$\ell_r := [, \dots,] : L^{\otimes r} \to L$$

of degree $r-2$ for $r \geq 1$ called brackets such that

(1) ℓ_k are skew-symmetric, that is,

$$[y_{\sigma(1)}, \dots, y_{\sigma(r)}] = \operatorname{sgn}(\sigma)\epsilon(\sigma)[y_{\sigma(1)}, \dots, y_{\sigma(r)}],$$

where $\operatorname{sgn}(\sigma)$ is the sign of σ.

(2) The generalized Jacobi identities are given by

$$\sum_{m+j=r+1}\sum_{\sigma}\operatorname{sgn}(\sigma)\epsilon(\sigma)(-1)^{m(j-1)}\ell_j(\ell_m(x_{\sigma(1)},\dots,y_{\sigma(m)}),x_{\sigma(m+1)},\dots,y_{\sigma(r)})=0,$$

where $\sigma\in S(m,r-m)$.

We follow [2] for this definition. Thus $\widetilde{\operatorname{Der}}(C,E;f)$ is defined as follows:

$$\widetilde{\operatorname{Der}}_i(C,E;f)=\begin{cases}\operatorname{Der}_i(C,E;f), & i>1,\\ \{\alpha\in\operatorname{Der}_1(C,E;f):\partial\alpha=0\}, & i=1.\end{cases}$$

Let $(C,d)=(\wedge W,d)$ be a Sullivan algebra and let $\alpha_1,\dots,\alpha_r\in\widetilde{\operatorname{Der}}(\wedge W,E;f)$ be f-derivations of respective degrees $m_1,\dots,m_r$. We define their bracket $[\alpha_1,\dots,\alpha_r]\in\widetilde{\operatorname{Der}}(\wedge W,E;f)$ of length r by

$$[\alpha_1,\dots,\alpha_r](w)=(-1)^\eta\sum\sum_{i_1,\dots,i_r}\epsilon f(w_1\dots\hat{w_{i_1}}\dots\hat{w_{i_r}}\dots w_j)\alpha_1(w_{i_1})\dots\alpha_r(w_{i_r}),$$

where $dw=\sum w_1\dots w_r$, $\eta=m_1+\cdots+m_{r-1}$ and ϵ is the suitable sign given by the Koszul convention. The desuspension defines linear maps ℓ_r for $r\geq 1$ each of degree $r-2$ on $s^{-1}\widetilde{\operatorname{Der}}(\wedge W,E;f)$ by

$$\ell_1(s^{-1}\alpha)=-s^{-1}\partial'\alpha,\ \ \ell_r(s^{-1}\alpha_1,\dots,s^{-1}\alpha_r)=(-1)^\beta s^{-1}[\alpha_1,\dots,\alpha_r],$$

where $\beta=\frac{r^2-r}{2}+\sum_{i=1}^{r-1}(r-i)|\alpha_i|$ [2]. It was shown in [2] that $(s^{-1}\operatorname{Der}(\wedge W,E;f),\ell_r)$ is an L_∞ model of maps$(M,N;h)$.

4. Preliminary results

Consider a map $\iota:\mathbb{H}P^m\hookrightarrow\mathbb{H}P^{m+r}$. In [17], the minimal Sullivan model of $\mathbb{H}P^m$ is given by $(\wedge(x_4,x_{4m+3}),d)$ where $dx_4=0$, $dx_{4m+3}=x_4^{m+1}$, and the minimal Sullivan model of $\mathbb{H}P^{m+r}$ is given by $(\wedge(y_4,y_{4(m+r)+3}),d)$ with $dy_4=0$, $dy_{4(m+r)+3}=y_4^{m+r+1}$. Moreover, the map $\mathbb{H}P^m\hookrightarrow\mathbb{H}P^{m+k}$ is modeled by

$$f:\wedge y_4/(y_4^{m+r+1})\to\wedge x_4/(x_4^{m+1}),$$

where $f(y_4)=x_4$. We have the following results.

Theorem 4.1. *Let $E=(\wedge(x_4,x_{4m+3}),d)$. Then $G_m(E)=\langle[x^*_{4m+3}]\rangle$.*

Proof. Consider $\operatorname{Der}(E,E;1)=\oplus_{i=0}^m\mathbb{Q}\alpha_{4i+3}\oplus\mathbb{Q}\alpha_4$, where α_4 is the derivation taking x_4 to one and α_{4i+3} is the derivation taking x_{4m+3} to x_4^{m-i} for $i=0,\dots,m$. Then $\delta\alpha_{4i+3}=0$ and $\delta\alpha_4=(m+1)\alpha_3$. Hence, for $1\leq i\leq m$, $[\alpha_{4i+3}]$ is nonzero in $H_*(\operatorname{Der}(E,E;1))$. Moreover, $\varepsilon_*(\alpha_{4i+3})=x^*_{4i+3}$. As $\mathbb{H}P^m$ is a finite CW-complex, then $G_{even}(E)=0$ (see [3, p. 379]). Hence, $G_m(E)=\langle[x^*_{4i+3}]\rangle$. □

Lemma 4.2. *Let $f:C=(\wedge(y_4,y_{4(m+r)+3}),d)\to\wedge x_4/(x_4^{m+1})=E$, where $f(y_4)=x_4$ and $f(y_{4(m+r)+3})=0$ be given. Then, an f-derivation θ_4 is a cycle.*

Proof. As $\theta_4(y_4) = 1$, then $\partial(\theta_4)(y_4) = 0$. Now it only remains to define θ_4 on $y_{4(m+r)+3}$ such that

$$d\theta_4(y_{4(m+r)+3}) - \theta_4(dy_{4(m+r)+3}) = 0.$$

Hence,

$$\begin{aligned} d\theta_4(y_{4(m+r)+3}) - \theta_4(dy_{4(m+r)+3}) &= d\theta_4(y_{4(m+r)+3}) - \theta_4(y_4^{m+r+1}) \\ & d\theta_4(y_{4(m+r)+3}) - (m+r+1)y_4^{m+r}. \end{aligned}$$

As the dimension of $\mathbb{H}P^m$ is $4m$ and $4m$ is less than $4(m+r)$ for $r \geq 1$, then $(m+r+1)y_4^{m+r}$ is boundary, that is, $(m+r+1)y_4^{m+r} = dt$. Define $\theta_4(y_{4(m+r)+3}) = t$. Moreover, $\partial\theta_4 = 0$. Therefore, θ_4 is nonzero in $H_*(\mathrm{Der}(C, E; f), \partial)$. $\square$

Theorem 4.3. *Let $f : C \to E$ be a Sullivan model of a map $\mathbb{H}P^m \hookrightarrow \mathbb{H}P^{m+r}$. Then, $G_*(C, E; f) = \langle [y_4^*], [y_{4(m+r)+3}^*] \rangle$.*

Proof. Define the derivation $\theta_{4(m+r)+3} = (y_{4(m+r)+3}, 1)$ in $\mathrm{Der}(C, E; f)$. Then $\partial\theta_{4(m+r)+3} = 0$. Moreover, $[\theta_{4(m+r)+3}]$ is nonzero in $H_*(\mathrm{Der}(C, E; f), \partial)$, and $[\theta_4]$ is nonzero in $H_*(\mathrm{Der}(C, E; f), \partial)$ by Lemma 4.2. Furthermore, $H(\varepsilon_*)([\theta_4]) = [y_4^*]$ and $H(\varepsilon_*)([\theta_{4(m+r)+3}]) = [y_{4(m+r)+3}^*]$. It then follows that $G_*(C, E; f) = \langle [y_4^*], [y_{4(m+r)+3}^*] \rangle$. $\square$

We note that for $r = 0$, it is easily verified that the model of $\mathrm{aut}_1 \mathbb{H}P^m = \mathrm{maps}(\mathbb{H}P^m, \mathbb{H}P^m, 1)$ has the rational homotopy type of the product $S^7 \times S^{11} \times \cdots \times S^{4n+3}$ (see Theorem 4.1). From now on, we assume $r \geq 1$, and we establish the following results (see [18] for another different proof).

Theorem 4.4. *The mapping space* $\mathrm{maps}(\mathbb{H}P^m, \mathbb{H}P^{m+r}, \iota)$ *is modeled by*

$$(\wedge(z_4, z_{4r+3}, \ldots, z_{4(m+r)+3}), d),$$

where $dz_4 = 0$ and $dz_{4r+3} = z_4^{r+1}, \ldots, dz_{4(m+r)+3} = z_4^{m+r+1}$.

Proof. Consider the map

$$f : C = (\wedge(y_4, y_{4(m+r)+3}), d) \to \wedge x_4/(x_4^{m+1}) = D.$$

Then by Theorem 4.3, a vector space $\widetilde{\mathrm{Der}}(C, E; f)$ is spanned by $\{\beta_4, \beta_{4r+4i-1},\ i = 1, \ldots, m+1\}$, where $\beta_{4r+4i-1} = (y_{4(m+r)+3}, y_4^{m-i+1})$ and $\beta_4 = (y_4, 1)$. Thus, an L_∞ model (L, ℓ_r) of $\mathrm{maps}(\mathbb{H}P^m, \mathbb{H}P^{m+r}, \iota)$ is spanned by $\langle s^{-1}\beta_4, s^{-1}\beta_{4r+4i-1},\ i = 1, \ldots, m+1 \rangle$. A straightforward calculation shows that the only nonzero brackets are as follows: $[\beta_4, \ldots, \beta_4] = \beta_{4r+4i-1}$, $i = 1, \ldots, m+1$. Hence, $\ell_j = 0$ for $j = 1, \ldots r$ and $\ell_{r+i}(s^{-1}\beta_4, \ldots, s^{-1}\beta_4) = \beta_{4r+4i-1}$.
Therefore,

$$C^\infty(L) = \wedge(z_4, z_{4r+3}, z_{4r+7}, \ldots, z_{4(m+r)+3}), d),$$

where $dz_4 = 0, dz_{4(r+i)+3} = z_4^{r+i+1}$ for $0 \leq i \leq m$. $\square$

Lemma 4.5. *Let $(\wedge V, d) = (\wedge(V_0 \oplus V_1), d)$ be a minimal Sullivan model of an F_0-space, where V is finite dimensional and $dV_0 = 0$, $dV_1 \subseteq \wedge V_0$. If $V_0^{even} = \mathbb{Q} < x_1, \ldots, x_n >$ and $V_1^{odd} = \mathbb{Q} < y_1, \ldots, y_r >$, then the generators $y_1 \ldots, y_r$ are Gottlieb elements, where the subscripts indicate the degrees.*

Proof. For $i \in \{1, \dots, r\}$, denote by θ_i the derivation of $\wedge V$ defined by $\theta_i(y_j) = \delta_{ij}$. A straightforward calculation shows that $\partial\theta_i(y_i) = 0$. Thus, the generators y_i are Gottlieb elements. $\square$

Proposition 4.6. *Let M be an F_0-space for which $\pi_*(M) \otimes \mathbb{Q}$ is finite dimensional, and let $E = (\wedge(V_0 \oplus V_1), d)$ be its minimal Sullivan model. Then $G_*(E)$ is generated by $< [y_1^*], \dots, [y_r^*] >$ as a vector space, where subscripts indicate the degrees.*

Proof. As $E = (\wedge V, d) = (\wedge(V_0 \oplus V_1), d)$ with $V_1^{\mathrm{odd}} = \mathbb{Q} < y_1, \dots, y_r >$, denote by θ_i the derivation of $\wedge V$ defined by $\theta_i(y_j) = \delta_{ij}$. It is easily verified that $\partial\theta_i(y_i) = 0$. Then, by Lemma 4.5, the generators $y_1, \dots, y_r$ are Gottlieb elements. Also, $[\theta_1], \dots, [\theta_r]$ are nonzero homology classes in $H_*(\mathrm{Der}(E, E; 1))$. It follows that $\varepsilon_*(\theta_1) = y_1^*, \dots, \varepsilon_*(y_r^*)$. Since M is a simply connected finite CW-complex, then $G_{\mathrm{even}}(E) = 0$ [3, Proposition 28.8]. Hence, $G_*(E)$ is generated by $< [y_1^*], \dots, [y_r^*] >$ as a vector space. $\square$

5. The main result

We now prove Theorem 1.1.

Proof of Theorem 1.1. By Theorem 4.4, the mapping space $\mathrm{maps}(\mathbb{H}P^m, \mathbb{H}P^{m+r}, \iota)$ is modeled by

$$(\wedge(z_4, z_{4r+3}, z_{4r+7}, \dots, z_{4(m+r)+3}), d),$$

where $dz_4 = 0, dz_{4(r+i)+3} = z_4^{r+i+1}$ for $0 \le i \le m$. The fibration $S^{4r+7} \to M \xrightarrow{p} \mathbb{H}P^r$ is modeled by

$$(\wedge(z_4, z_{4r+3}), d) \to (\wedge(z_4, z_{4r+3}) \otimes \wedge z_{4r+7}, D),$$

where $dz_4 = 0$, $dz_{4r+3} = z_4^{r+1}$, $Dz_4 = dz_4$, $Dz_{4r+3} = dz_{4r+3}$, $Dz_{4r+7} = z_4^{r+2}$. Since Dz_{4r+7} is a coboundary in $H^*(\wedge(z_4, z_{4r+3}), d)$, then, p is a trivial fibration (see [5]). Hence the cdgas

$$(C, d) = (\wedge(z_4, z_{4r+3}, z_{4r+7}), d),$$

where $dz_4 = 0, dz_{4r+3} = z_4^{r+1}$, $dz_{4r+7} = z_4^{r+2}$, and

$$(\wedge(z_4, z_{4r+3}) \otimes \wedge z_{4r+7}, D),$$

where $Dz_4 = dz_4$, $Dz_{4r+3} = dz_{4r+3}$, $Dz_{4r+7} = 0$, are isomorphic. Hence the cdga (C, d) is a model of $\mathbb{H}P^r \times S^{4r+7}$. It follows from an induction argument that $\mathrm{maps}(\mathbb{H}P^m, \mathbb{H}P^{m+r}, \iota)$ has the rational homotopy type of $\mathbb{H}P^r \times S^{4r+7} \times \cdots \times S^{4(m+r)+3}$. $\square$

On one hand, we have the following result.

Corollary 5.1. *The mapping space* $\mathrm{maps}(\mathbb{H}P^m, \mathbb{H}P^{m+r}, \iota)$ *is formal.*

On the other hand, consider the inclusion $\iota : \mathbb{H}P^m \to \mathbb{H}P^{m+r}$ and the corresponding model $f : C = (\wedge(y_4, y_{4(m+r)+3}), d) \to (\wedge(x_4, x_{4m+3}), d) = E$. Forgetting the desuspension, a model of the inclusion $\iota_* : \mathrm{aut}_1 \mathbb{H}P^m \to \mathrm{maps}(\mathbb{H}P^m, \mathbb{H}P^{m+r}, \iota)$ is given by

$$f^* : \mathrm{Der}(E, E; 1) \to \mathrm{Der}(C, E; f).$$

The map f^* is characterized as follows when $r > m$.

Theorem 5.2. *If $r > m$, then the induced map*

$$f^* : \mathrm{Der}(E, E; 1) \to \mathrm{Der}(C, E; f)$$

is homotopy trivial.

Proof. Recall that $L = \mathrm{Der}(E, E; 1) = \oplus_{i=0}^{m} \mathbb{Q}\alpha_{4i+3} \oplus \mathbb{Q}\alpha_4$, where $\alpha_4 = (x_4, 1)$ and $\alpha_{4i+3} = (x_{4m+3}, x_4^{m-i})$ for $i = 0, \ldots, m$. Then $\delta\alpha_{4i+3} = 0$ and $\delta\alpha_4 = (m+1)\alpha_3$. Therefore,

$$\pi_*(\mathrm{aut}_1 \mathbb{H}P^m) \otimes \mathbb{Q} = H_*(L, \delta)) = \langle [\alpha_7], \ldots, [\alpha_{4m+3}] \rangle.$$

Hence, $\mathrm{aut}_1 \mathbb{H}P^m$ has the rational homotopy type of $S^7 \times S^{11} \times \cdots \times S^{4m+3}$. Let

$$L' = (\mathrm{Der}(C, E; f), \partial) = (\langle \oplus_{i=r}^{r+m} \mathbb{Q}\beta_{4i+3} \oplus \mathbb{Q}\beta_4 \rangle, \partial),$$

for $i = r, r+1, \ldots, r+m$. The mapping $f^* : L \to L'$ is defined by $f^*(\alpha_4)$, $f^*(\alpha_{4i+3}) = 0$ for $i < r$, and $f^*(\alpha_{4i+3}) = \beta_{4i+3}$ for $i \geq r$. If $r > m$, then $f^*(\alpha_4) = \beta_4$ and zero elsewhere. Furthermore,

$$C^\infty(s^{-1}L) = (\wedge(x_4, x_3, \ldots, x_{4i-1}, \ldots, x_{4m+3}), d),$$

where $dx_4 = 0$ and $dx_{4i-1} = x_4^i$ for $i = 1, \ldots, m+1$. Likewise,

$$C^\infty(s^{-1}L') = (\wedge(y_4, y_{4r+3}, \ldots, y_{4(m+r)+3}), d),$$

where $dy_4 = 0$ and $dy_{4i+3} = x_4^{i+1}$ for $i = r, r+1, \ldots, m+r$. As $C^\infty(s^{-1}L')$ is quasi-isomorphic to

$$(\wedge(w_4, w_{4r+3}), d) \otimes (\wedge(w_{4r+7}, \ldots, w_{4(m+r)+3}), 0),$$

where $dw_4 = 0$, $dw_{4r+3} = w_4^{r+1}$, and $C^\infty(s^{-1}L)$ is quasi-isomorphic to

$$(\wedge(z_7, \ldots, z_{4m+3}), 0),$$

and the induced map

$$\bar{\phi} : (\wedge(w_4, w_{4r+3}, w_{4r+7}, \ldots, w_{4(m+r)+3}), d) \to (\wedge(z_7, \ldots, z_{4m+3}), 0)$$

between minimal models is zero. □

Definition 5.3. Let $f : C \to E$ be a map, where C and E are differential graded vector spaces. The mapping cone of f, denoted $Rel_*(f)$ (see, for example, [21, 13]) is defined by $Rel_m(f) = C_{m-1} \oplus E_m$ for all $m > 1$, and $D(x, y) = (-d_C(x), f(x) + d_E(y))$. The chain maps $J : E_m \to Rel_m(f)$ and $P : Rel_m(f) \to C_{m-1}$ are defined by $J(w) = (0, w)$ and $P(x, y) = x$, respectively. These yield a short exact sequence of chain complexes

$$0 \to E_* \xrightarrow{J} Rel_*(f) \xrightarrow{P} C_{*-1} \to 0,$$

a long exact homology sequence of f

$$\cdots \to H_{m+1}(Rel(f)) \xrightarrow{H(P)} H_m(C) \xrightarrow{H(f)} H_m(E) \xrightarrow{H(J)} H_m(Rel(f)) \to \cdots,$$

and a connecting homomorphism $H(f)$.

Following [13], there is a commutative diagram;

$$\begin{tikzcd} \mathrm{Der}(E,E;1) \arrow[r, "f^*"] \arrow[d, "\varepsilon_*"'] & \mathrm{Der}(C,E;f) \arrow[d, "\varepsilon_*"] \\ \mathrm{Der}(E,\mathbb{Q};\varepsilon) \arrow[r, "\widehat{f^*}"] & \mathrm{Der}(C,\mathbb{Q};\varepsilon), \end{tikzcd}$$

where ε is the augmentation of either C or E. The homology ladder for $m \geq 2$, is given by

$$\begin{tikzcd} \cdots \to H_{m+1}(Rel(f^*)) \arrow[r, "H(P)"] \arrow[d, "H(\varepsilon_*,\varepsilon_*)"'] & H_m(\mathrm{Der}(E,E;1)) \arrow[r, "H(f^*)"] \arrow[d, "H(\varepsilon_*)"'] & H_m(\mathrm{Der}(C,E;f)) \to \cdots . \arrow[d, "H(\varepsilon_*)"'] \\ \cdots \to H_{m+1}(Rel(\widehat{f}^*)) \arrow[r, "H(\widehat{P})"] & H_m(\mathrm{Der}(E,\mathbb{Q};\varepsilon)) \arrow[r, "H(\widehat{\phi^*})"] & H_m(\mathrm{Der}(C,\mathbb{Q};\varepsilon)) \to \cdots \end{tikzcd}$$

Thus, the mth relative evaluation subgroup of f is defined as follows:

$$G^{rel}_m = \mathrm{Im}\{H(\varepsilon_*,\varepsilon_*) : H_m(Rel(f^*)) \to H_m(Rel(\widehat{f}^*))\}.$$

The G-sequence of the map $f : C \to E$ is given by the sequence

$$\cdots \overset{H(\widehat{J})}{\to} G^{rel}_{m+1}(C,E;\phi) \overset{H(\widehat{P})}{\to} G_m(E) \overset{H(\widehat{f^*})}{\to} G_m(C,E;f) \overset{H(\widehat{J})}{\to} \cdots ,$$

which terminates in $G_2(C,E;f)$. Moreover, in [13, Theorem 3.5], it was shown that this can be applied to the Sullivan model $f : C \to E$ of the map $h : M \to N$.

We are now in position to prove Theorem 1.2 .

Proof of Theorem 1.2. It follows from Theorems 4.1 and 4.3 that

$$G_m(E) = \langle [x^*_{4m+3}] \rangle$$

and that

$$G_*(C,E;f) = \langle [y^*_4], [y^*_{4(m+r)+3}] \rangle.$$

We begin with the case where $r > m$. Let α_4, $\alpha_{4i+3} \in \mathrm{Der}(E,E;1)$ and β_4, $\beta_{4i+3} \in \mathrm{Der}(C,E;f)$ be defined as above. Then, $f^*(\alpha_4) = \beta_4$ and $f^*(\alpha_{4i+3}) = 0$. Furthermore, $D(\alpha_4, 0) = (0, \beta_4)$, $D(\alpha_{4i+3}, 0) = (0, 0)$, and $D(0, \beta_4) = 0 = D(0, \beta_{4i+3})$. Therefore, $[(\alpha_{4i+3}, 0)]$ and $[(0, \beta_{4i+3})]$ are nonzero in $H_*(Rel(f^*))$. We conclude that

$$G^{rel}_*(C,E;f) = \langle [(x^*_{4m+3}, 0)], [(0, y^*_{4(m+r)+3})] \rangle.$$

Hence, the G-sequence reduces to the fragments

$$0 \to G^{rel}_{2m+1}(C,E;f) \underset{\simeq}{\overset{H(P)}{\to}} G_{2m+1}(E) \to 0,$$

$$0 \to G_{4(m+r)+3}(C,E;f) \underset{\simeq}{\overset{H(J)}{\to}} G^{rel}_{4(m+r)+3}(C,E;f) \to 0,$$

and terminates with

$$0 \to G_4(C,E;f) \to 0.$$

As $G_4(C,E;f) \cong \mathbb{Q}$, we conclude that the last fragment of the G-sequence is not exact. Moreover, if $r \leq m$, then $f^*(\alpha_{4i+3}) = \beta_{4i+3}$. Thus, $D(\alpha_{4i+3}, 0) =$

$(0, \beta_{4i+3})$. Hence, $[(x^*_{4m+3}, 0)] \in H_*(Rel(f^*))$ is not in the image of $H_*(\varepsilon_*, \varepsilon_*)$. The G-sequence reduces to the fragment

$$0 \to G_{2m+1}(E) \to 0,$$

which is not exact. □

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