Khayyam J. Math. 10 (2024), no. 1, 90-96
DOI: 10.22034/KJM.2024.406773.2932

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

APPROXIMATE BIPROJECTIVITY AND PSEUDO-CONTRACTIBILITY OF CERTAIN SEMIGROUP ALGEBRAS

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Communicated by A.R.K. Mirmostafaee

ABSTRACT. In this paper, we study the approximate biprojectivity and the pseudo-contractibility of certain semigroup algebras and their second dual. In particular, we study the approximate biprojectivity of $l^1(S)^{**}$, where S is a semigroup. In continuation, we give some examples that show that some of the existing results in the literature are not correct as presented, and in addition, we provide more modifications.

1. INTRODUCTION AND PRELIMINARIES

The concepts of pseudo-amenability and pseudo-contractibility for Banach algebras have been introduced by Ghahramani and Zhang in [7]. They studied the pseudo-amenability and the pseudo-contractibility of Banach algebras associated to locally compact groups, such as group algebras, measure algebras, and Segal algebras. Essmaili et al. [5] investigated the above notions for semigroup algebras over an inverse semigroup with uniformly locally finite idempotent set. They also showed that if $l^1(S)$ is pseudo-contractible and S has a left or right unit, then S is finite.

One of the most important notions pertinent to amenability in the theory of homological Banach algebras is biprojectivity, introduced by Helemskii [8]. Recall that a Banach algebra $\mathcal{A}$ is called *biprojective* if there exists a bounded $\mathcal{A}$-bimodule morphism $m : \mathcal{A} \longrightarrow \mathcal{A}\hat{\otimes}\mathcal{A}$ such that m is a right inverse for $\pi : \mathcal{A}\hat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$, the product morphism which specified by $\pi(a \otimes b) = ab$; some properties of biprojective semigroup algebras were investigated in [11]. Recall

Date: Received: 12 July 2023; Revised: 3 February 2024; Accepted: 10 February 2024.
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2020 *Mathematics Subject Classification.* Primary 46H20; Secondary 43A20.
Key words and phrases. Approximate biprojectivity, contractibility, inverse semigroup.

that the biprojectivity of second dual of a Banach algebra was studied in [10]. The concepts of approximate biprojectivity have been proposed and studied by Zhang [13]. Sahami and Pourabbas proved that, for a semigroup S, the approximate biprojectivity of $l^1(S)$ implies finiteness of S, when S has a left or right unit and $Z(S) \neq \emptyset$, where $Z(S)$ is the centralizer of S; see [12, Proposition 3.1] for more details. Essmaili and Medghalchi [6] showed that for a certain class of inverse semigroups, the biprojectivity of $l^1(S)^{**}$ is equivalent to the biprojectivity of $l^1(S)$. In fact, they proved that when E (the set of idempotents of S) is finite, the biprojectivity of $l^1(S)$ and its second dual are equivalent. For module version of the biprojectivity of Banach algebras, we refer to [1].

In the current work, we first present some counterexamples to show that the proof of [5, Corollary 2.10] has a gap and correct its proof. On the other hand, the proof of [12, Proposition 3.1] is based on the proof of [5, Corollary 2.10] and has a similar gap. We bring an example and prove the same result for inverse semigroups under weaker conditions. Moreover, we study approximate the biprojectivity of $l^1(S)^{**}$. More precisely, we show that for a semigroup S, when $l^1(S)$ has a central bounded approximate identity, the approximate biprojectivity of $l^1(S)^{**}$ implies the regularity of S and the finiteness of E. In the case that S is an inverse semigroups, it must be finite.

2. Main results

We first recall some background definitions and notations in the Banach algebras setting, and then we investigate the pseudo-contractibility and approximate biprojectivity of miscellaneous semigroup algebras. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras. We denote the projective tensor product of $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A}\hat{\otimes}\mathcal{B}$. It is known that the Banach algebra $\mathcal{A}\hat{\otimes}\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule with the following actions:

$$a \cdot (b \otimes c) = ab \otimes c, (b \otimes c) \cdot a = b \otimes ca \qquad (a, b, c \in \mathcal{A}).$$

Definition 2.1. (i) A Banach algebra $\mathcal{A}$ is said to be *pseudo-amenable* if there is a net $(m_\alpha) \subseteq \mathcal{A}\hat{\otimes}\mathcal{A}$, which is called an *approximate diagonal* for A, such that for each $a \in \mathcal{A}$,

$$a \cdot m_\alpha - m_\alpha \cdot a \to 0 \textit{ and } \pi(m_\alpha)a \to a;$$

(ii) A Banach algebra $\mathcal{A}$ is said to be *pseudo-contractible* if there is a net (m_α) in $\mathcal{A}\hat{\otimes}\mathcal{A}$, which is called an *central approximate diagonal* for $\mathcal{A}$, such that for each $a \in \mathcal{A}$,

$$a \cdot m_\alpha = m_\alpha \cdot a \textit{ and } \pi(m_\alpha)a \to a.$$

Clearly, every pseudo-amenable and pseudo-contractible Banach algebra has a left approximate identity.

Let $\mathcal{A}$ be a Banach algebra, and let X be a Banach $\mathcal{A}$-bimodule. A bounded linear map $D : \mathcal{A} \longrightarrow X$ is called a *derivation* if

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad (a, b \in \mathcal{A}).$$

For each $x \in X$, we define the map $ad_x : \mathcal{A} \longrightarrow X$; $a \mapsto a \cdot x - x \cdot a$ for all $a \in \mathcal{A}$. It is easily checked that ad_x is a derivation. Derivations of this form are called *inner derivations*.

Definition 2.2 (see [13])**.** A Banach algebra $\mathcal{A}$ is called *approximately biprojective* if there exists a net $m_\alpha : \mathcal{A} \longrightarrow \mathcal{A}\hat{\otimes}\mathcal{A}$ such that m_α's are continuous $\mathcal{A}$-bimodule morphism and $\pi \circ m_\alpha(a) \to a$ for all $a \in \mathcal{A}$.

Definition 2.3. A Banach algebra $\mathcal{A}$ is *contractible* if for each Banach $\mathcal{A}$-bimodule X, every continuous derivation $D : \mathcal{A} \longrightarrow X$ is inner.

For a Banach algebra $\mathcal{A}$, $\mathcal{A}^{\#} = \mathcal{A} \oplus \mathbb{C}$, the unitization of $\mathcal{A}$, is a unital Banach algebra that contains $\mathcal{A}$ as a closed ideal. Merging [7, Theorem 2.4 and Proposition 3.8], similar to [7, Theorem 2.4], one can obtain the following Theorem.

Theorem 2.4. *Let $\mathcal{A}$ be a Banach algebra. Then, the following statements are equivalent:*

(i) *$\mathcal{A}^{\#}$ is approximately biprojective;*
(ii) *$\mathcal{A}$ is approximately biprojective and has an identity;*
(iii) *$\mathcal{A}$ is contractible.*

Example 2.5. The Banach algebra $\ell^1 := l^1(\mathbb{N})$ under pointwise multiplication is pseudo-contractible and so is approximately biprojective. Since ℓ^1 does not have an identity, it follows from Theorem 2.4 that $\ell^{1\#}$ is not approximate biprojective.

Proposition 2.6. *Let $\mathcal{A}$ be a Banach algebra with a left identity (resp., right identity) and a right (resp., left) approximate identity. Then, it has an identity.*

Proof. Assume that $a \in \mathcal{A}$, that e is a left identity and that (e_α) is a right approximate identity for $\mathcal{A}$. We have

$$a \cdot e = \lim_\alpha (a \cdot e) \cdot e_\alpha = \lim_\alpha a \cdot (e \cdot e_\alpha) = \lim_\alpha a \cdot e_\alpha = a.$$

Hence, e is a right identity for $\mathcal{A}$ and so it has an identity. □

Proposition 2.7. *Let $\mathcal{A}$ be a pseudo-contractible Banach algebra with a left identity. Then, it is biprojective.*

Proof. Our assumption implies that $\mathcal{A}$ has a central approximate identity. By Proposition 2.6, it has an identity, and therefore $\mathcal{A}$ is contractible (see [7, Theorem 2.4]). Now, a direct consequence of [2, Theorem 2.8.48] shows that $\mathcal{A}$ is biprojective. □

A semigroup S is *regular* if for each $s \in S$, there exists $t \in S$ with $sts = s$. An inverse semigroup is a semigroup S so that, for each $s \in S$, there exists a unique element $s^* \in S$ such that $ss^*s = s$ and $s^*ss^* = s^*$. The element s^* is termed the inverse of s. The set $E(S)$ (or briefly, E) of idempotents of S is a commutative subsemigroup; it is ordered by

$$e \leq f \iff ef = e.$$

With this ordering $E(S)$ is a meet semilattice with the meet given by the product; see [9, Theorem 5.1.1]. We recall that a semigroup S is a *semilattice* if S is commutative and $E = S$. The order on E extends to S as the so-called natural partial order by putting $s \leq t$ if $s = et$ for some idempotent e (or equivalently $s = tf$ for some idempotent f). This is equivalent to $s = ts^*s$ or $s = ss^*t$. If $e \in E$, then the set $G_e = \{s \in S | ss^* = s^*s = e\}$ is a group, called the *maximal subgroup* of S at e.

Let S be an inverse semigroup. For every $x \in S$, we denote $(x] = \{y \in S | y \leq x\}$. Moreover, S is called *locally finite* (resp., *uniformly locally finite*) if for each $x \in S$, $|(x]| < \infty$ (resp., $\sup\{|(x]| : x \in S\} < \infty$).

Remark 2.8. It is proved in [5, Corollary 2.10] that if $l^1(S)$ is pseudo-contractible and S has a left or right unit, then S is finite. The argument given in [5] is not correct as it stands. In fact, the authors have used the existence of an element $m \in l^1(S)\hat{\otimes} l^1(S)$ such that

$$\delta_s \cdot m = m \cdot \delta_s = m,$$

for all $s \in S$. The last equality shows that $\delta_s \cdot \pi(m) = \pi(m) \cdot \delta_s = \pi(m)$, and hence

$$\pi(m)(\delta_s) = \pi(m)(\delta_{ss^*s}) = \pi(m) \cdot \delta_s(\delta_{ss^*}) = \pi(m)(\delta_{ss^*}), \tag{2.1}$$

for all $s \in S$. On the other hand, for each $e, f \in E$, we get

$$\begin{aligned} \pi(m)(\delta_e) &= (\pi(m) \cdot \delta_f) \cdot (\delta_e) \\ &= \pi(m)(\delta_{fe}) \\ &= \pi(m)(\delta_{ef}) \\ &= (\pi(m) \cdot \delta_e) \cdot \delta_f \\ &= \pi(m)(\delta_f). \end{aligned} \tag{2.2}$$

It follows from (2.1) and (2.2) that $\pi(m)$ is a constant function on $l^1(S)$. Since $\pi(m) \neq 0$ and $\pi(m) \in l^1(S)$, S is finite. In other words, if the presented argument is correct as it stands, for each inverse semigroup S, the pseudo-contractibility of $l^1(S)$ should imply that S is finite. This is while there are plenty of known examples of infinite inverse semigroups with pseudo-contractible semigroup algebra. For a typical example, let $S = \mathbb{Z}$ (set of integers) for which the multiplication is defined by

$$m \star n = \begin{cases} m & if\ m = n, \\ 0 & if\ m \neq n. \end{cases}$$

By [5, Theorem 2.4], $l^1(S)$ is pseudo-contractible while S is not finite.

We correct the proof of [5, Corollary 2.10] as follows.

Proposition 2.9. *Let S be a semigroup such that S has a left or right unit. Then, the following statements are equivalent:*

(i) *$l^1(S)$ is pseudo-contractible;*
(ii) *S is finite.*

Proof. (i)⇒(ii) Suppose that S has a left unit. Thus, $l^1(S)$ has a left identity. Since $l^1(S)$ is pseudo-contractible, it has a central approximate identity and by

Proposition 2.6, $l^1(S)$ has an identity. Hence, [7, Theorem 2.4] implies that $l^1(S)$ is contractible. It follows from [3, Theorem 2] that S is regular and E is finite . Now, [4, Theorem 3.5] can be applied to show that S is finite. For the right case, the proof is similar.
(2)⇒(1) It is clear. □

Example 2.10. Let $S = \left\{ \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} : a, b \in \mathbb{C} \right\}$. With the matrix multiplication, S is a semigroup. We claim that $l^1(S)$ is biprojective. Clearly, $s_1 = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ is a right unit for S. Define $m : l^1(S) \longrightarrow l^1(S)\hat{\otimes} l^1(S)$ by

$$m\left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\right) = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad (a, b \in \mathbb{C}).$$

Then, for each $a, b, c, d \in \mathbb{C}$, we have

$$\begin{aligned} m\left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \star \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}\right) &= \left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \star \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}\right) \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & ad \\ 0 & bd \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

On the other hand,

$$\begin{aligned} m\left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\right) \cdot \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} &= \left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right) \cdot \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} \\ &= \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \otimes \begin{pmatrix} 0 & d \\ 0 & d \end{pmatrix} = d \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & ad \\ 0 & bd \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Thus, $m\left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \star \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}\right) = m\left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\right) \cdot \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}$. Moreover,

$$\begin{aligned} \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \cdot m\left(\begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}\right) &= \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \cdot \left(\begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right) \\ &= \begin{pmatrix} 0 & ad \\ 0 & bd \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \\ &= m\left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \star \begin{pmatrix} 0 & c \\ 0 & d \end{pmatrix}\right). \end{aligned}$$

In addition,

$$\pi \circ m\left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}\right) = \pi\left(\begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & b \end{pmatrix}.$$

Therefore, $l^1(S)$ is biprojective.

Remark 2.11. Using [12, Proposition 3.1], Sahami and Pourabbas in [12, Example 3.3] showed that the semigroup algebra $l^1(S)$ defined in Example 2.10 is not approximately biprojective while we observed that it is biprojective. Since

$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in Z(S)$, the example above is a counterexample that shows that [12, Proposition 3.1] cannot be correct as stated.

Here, we correct the proof of [12, Proposition 3.1] for inverse semigroups.

Theorem 2.12. *Let S be an inverse semigroup. If $l^1(S)$ is approximately biprojective and S has a left or right unit, then S is finite.*

Proof. Suppose that e is a left unit for the inverse semigroup S. We claim that e is a right unit for S. For every $s \in S$, we have

$$\begin{aligned} se &= (se)(se)^*(se) = (se)(es^*)(se) = se\big((es^*)(se)\big) \\ &= s(es^*)(se) = s(es^*s) = ss^*s = s. \end{aligned}$$

Hence, e is a right unit and so a unit for S. Thus, $l^1(S)$ is approximately biprojective and $l^1(S)$ has an identity. By [7, Proposition 3.8], $l^1(S)$ is pseudo-contractible. The result now follows from Proposition 2.9. □

Remark 2.13. When S is an infinite right zero semigroup, obviously, S has a left unit. Moreover, $l^1(S)$ is biprojective by [6, Proposition 3.1] and so is approximate biprojective. This shows that Theorem 2.12 does not hold for an arbitrary semigroup.

Example 2.14. Let S be a bicyclic semigroup. That is, $S = \{p^m q^n : m, n \geq 0\}$ with the multiplication

$$(p^m q^n)(p^s q^t) = p^{m-n+\max\{n,s\}} q^{t-s+\max\{n,s\}}.$$

It is clear that S has an identity and by Theorem 2.12, $l^1(S)$ is not approximate biprojective.

Recall from [11] that there exists an equivalence relation D on an inverse semigroup S such that sDt if and only if there exists $x \in S$ such that $ss^* = xx^*$ and $t^*t = x^*x$. We denote by $\{D_\lambda : \lambda \in \Lambda\}$ the collection of D-classes.

Remember that a Banach algebra $\mathcal{A}$ is called Arens regular if both the first and second Arens products are the same; for more details, we refer to [2].

Theorem 2.15. *Let S be a semigroup such that $l^1(S)$ is Arens regular and has a bounded approximate identity. If $l^1(S)^{**}$ is approximately biprojective, then*

(i) *S is regular and E is finite;*
(ii) *when S is an inverse semigroup, S is finite.*

Proof. (i) Suppose that (e_α) is bounded approximate identity for $l^1(S)$. We may assume that G is a weak* cluster point of (e_α). It is obvious that G is a unit element for $l^1(S)^{**}$. By the hypothesis that $l^1(S)^{**}$ is approximately biprojective, it can be concluded from [7, Proposition 3.8 and Theorem 2.4] that $l^1(S)^{**}$ is contractible. This shows that $l^1(S)$ is amenable, S is regular and E is finite.
(ii) By (i), $l^1(S)$ is amenable. Then $l^1(S)$ is biflat and so S is uniformly locally finite. It is well known that the contractibility of a Banach algebra is equivalent to its biprojectivity and being unital (see [2, Theorem 2.8.48]). It is shown in the first part that $l^1(S)^{**}$ is contractible, and hence $l^1(S)^{**}$ is biprojective. It follows

from [6, Theorem 3.9] that $l^1(S)$ is biprojective, and by [11, Theorem 3.7], each maximal subgroup of S is finite. Now, the finiteness of E necessitates that S is also finite. □

Remark 2.16. Note that when $S = \mathbb{Z}$ as in Remark 2.8, S is a uniformly locally finite semilattice and so $l^1(S)$ is biprojective. Thus, by [6, Corollary 3.11], $l^1(S)^{**}$ is biprojective and hence is approximately biprojective while S is not finite. This example shows that the hypothesis the bounded central approximate identity cannot be removed from Theorem 2.15.

Corollary 2.17. *Let S be an infinite Arens regular semillatice such that $l^1(S)$ has a bounded approximate identity. Then, $l^1(S)^{**}$ is not approximate biprojective.*

Acknowledgement. The authors sincerely thank the anonymous reviewers for their careful reading and constructive comments that improved the manuscript substantially.

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