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SHARP BOUNDS FOR THE SECOND HANKEL DETERMINANT OF LOGARITHMIC COEFFICIENTS FOR PARABOLIC STARLIKE AND UNIFORMLY CONVEX FUNCTIONS OF ORDER α

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ABSTRACT. Let $\mathcal{A}$ denote the class of analytic functions f in the open unit disk $\mathbb{U}$ normalized by $f(0) = f'(0) - 1 = 0$, and let $\mathcal{S}$ be the class of all functions $f \in \mathcal{A}$ that are univalent in $\mathbb{U}$. For a function $f \in \mathcal{S}$, the logarithmic coefficients δ_n $(n = 1, 2, 3, \ldots)$ are defined by

$$\log \frac{f(z)}{z} = 2 \sum_{n=1}^{\infty} \delta_n z^n \qquad (z \in \mathbb{U}).$$

For $0 \leq \alpha < 1$, let $\mathcal{S}_p(\alpha)$ and $\mathcal{UCV}(\alpha)$ denote the classes of functions $f \in \mathcal{A}$ such that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < (1 - 2\alpha) + \Re\left(\frac{zf'(z)}{f(z)} \right) \qquad (z \in \mathbb{U})$$

and

$$\left| \frac{zf''(z)}{f'(z)} \right| < 2(1 - \alpha) + \Re\left(\frac{zf''(z)}{f'(z)} \right) \qquad (z \in \mathbb{U}),$$

respectively. In the present paper, we determine the sharp upper bound for $|\delta_n|$ $(n = 1, 2, 3, \ldots)$ of functions f belonging to the classes $\mathcal{S}_p(\alpha)$. Also, we obtain upper bounds for $|\delta_n|$ $(n = 1, 2, 3)$ of functions belonging to the class $\mathcal{UCV}(\alpha)$.

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1. Introduction

Let $\mathbb{R} = (-\infty, \infty)$ be the set of real numbers, let $\mathbb{C}$ be the set of complex numbers, and let

$$\mathbb{N} := \{1, 2, 3, \ldots\} = \mathbb{N}_0 \backslash \{0\}$$

be the set of positive integers.

Assume that $\mathcal{H}$ is the class of analytic functions in the open unit disc

$$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\},$$

and let the class $\mathcal{P}$ be defined by

$$\mathcal{P} = \{p \in \mathcal{H} : p(0) = 1 \quad \text{and} \quad \Re(p(z)) > 0 \ (z \in \mathbb{U})\}.$$

For two functions $f, g \in \mathcal{H}$, we say that the function f is subordinate to g in $\mathbb{U}$ and write

$$f(z) \prec g(z) \qquad (z \in \mathbb{U})$$

if there exists a Schwarz function

$$\omega \in \Omega := \{\omega \in \mathcal{H} : \omega(0) = 0 \quad \text{and} \quad |\omega(z)| < 1 \ (z \in \mathbb{U})\}$$

such that

$$f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Rightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence relation:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ consisting of functions f normalized by

$$f(0) = f'(0) - 1 = 0.$$

Each function $f \in \mathcal{A}$ can be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in \mathbb{U}). \tag{1.1}$$

We also denote by $\mathcal{S}$ the class of all functions in the normalized analytic function class $\mathcal{A}$ that are univalent in $\mathbb{U}$.

Definition 1.1. A function $f \in \mathcal{A}$ is said to be starlike of order α $(0 \leq \alpha < 1)$ if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}).$$

We say that f is in the class $\mathcal{S}^*(\alpha)$ for such functions.

In particular, we set $\mathcal{S}^*(0) = \mathcal{S}^*$ for the class of starlike functions in the open unit disk $\mathbb{U}$. Recall that $\mathcal{S}^*(\alpha) \subset \mathcal{S}^* \subset \mathcal{S}$.

Definition 1.2 ([5])**.** A function $f \in \mathcal{A}$ is said to be parabolic starlike of order α $(0 \leq \alpha < 1)$ if and only if

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < (1-2\alpha) + \Re\left(\frac{zf'(z)}{f(z)}\right) \qquad (z \in \mathbb{U}).$$

We say that f is in the class $\mathcal{S}_p(\alpha)$ for such functions.

Equivalently,

$$f(z) \in \mathcal{S}_p(\alpha) \Leftrightarrow \frac{zf'(z)}{f(z)} \in \Omega_\alpha \qquad (z \in \mathbb{U}),$$

where Ω_α denotes the parabolic region in the right half-plane

$\Omega_\alpha = \left\{w = u + iv : v^2 < 4(1-\alpha)(u-\alpha)\right\} = \{w : |w-1| < (1-2\alpha) + \Re(w)\}.$

From its definition, it is clear that the class $\mathcal{S}_p(\alpha)$ is contained in the class $\mathcal{S}^*(\alpha)$.

Definition 1.3 ([15])**.** A function $f \in \mathcal{A}$ is said to be uniformly convex of order α $(0 \leq \alpha < 1)$ if and only if

$$\left|\frac{zf''(z)}{f'(z)}\right| < 2(1-\alpha) + \Re\left(\frac{zf''(z)}{f'(z)}\right) \qquad (z \in \mathbb{U}).$$

We say that f is in the class $\mathcal{UCV}(\alpha)$ for such functions.

Lee [15] showed that

$$f \in \mathcal{UCV}(\alpha) \Leftrightarrow zf' \in \mathcal{S}_p(\alpha). \tag{1.2}$$

In particular, we set $\mathcal{S}_p(1/2) = \mathcal{S}_p$ for the class of parabolic starlike functions introduced by Ronning [23], and $\mathcal{UCV}(1/2) = \mathcal{UCV}$ for the class of uniformly convex functions.

Ali and Singh [5] showed that the normalized Riemann mapping function $q_\alpha(z)$ from the open unit disk $\mathbb{U}$ onto Ω_α is given by

$$q_\alpha(z) = 1 + \frac{4(1-\alpha)}{\pi^2}\left[\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right]^2 := 1 + \sum_{n=1}^{\infty} D_n z^n \qquad (z \in \mathbb{U}). \tag{1.3}$$

The branch of $\sqrt{z}$ is chosen such that $\Im\sqrt{z} \geq 0$. Using the expansion of

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n \qquad (z \in \mathbb{U}),$$

we get

$$\left[\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right]^2 = 4z + \frac{8}{3}z^2 + \frac{92}{45}z^3 + \cdots. \tag{1.4}$$

From the above equalities (1.3) and (1.4), we obtain

$$q_\alpha(z) = 1 + \frac{16(1-\alpha)}{\pi^2}z + \frac{32(1-\alpha)}{3\pi^2}z^2 + \frac{368(1-\alpha)}{45\pi^2}z^3 + \cdots = 1 + \sum_{n=1}^{\infty} D_n z^n, \tag{1.5}$$

where

$$D_n = \frac{16(1-\alpha)}{n\pi^2}\sum_{j=0}^{n-1}\frac{1}{2j+1} \qquad (n \in \mathbb{N}). \tag{1.6}$$

Lemma 1.4 ([16]). *If $f \in \mathcal{S}_p(\alpha)$, then*

$$\frac{zf'(z)}{f(z)} \prec q_\alpha(z) \qquad (z \in \mathbb{U}),$$

where q_α is given in (1.3).

For a function $f \in \mathcal{S}$, given by (1.1), the logarithmic coefficients δ_n $(n \in \mathbb{N})$ are defined by

$$F_f(z) := \log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \delta_n z^n \qquad (z \in \mathbb{U}), \tag{1.7}$$

and play a central role in the theory of univalent functions. Note that, by differentiating (1.7) and equating coefficients, we have

$$\delta_1 = \frac{1}{2}a_2, \tag{1.8}$$

$$\delta_2 = \frac{1}{2}\left(a_3 - \frac{1}{2}a_2^2\right), \tag{1.9}$$

$$\delta_3 = \frac{1}{2}\left(a_4 - a_2a_3 + \frac{1}{3}a_2^3\right). \tag{1.10}$$

For the whole class $\mathcal{S}$, the sharp estimates of single logarithmic coefficients are known only for δ_1 and δ_2, namely,

$$|\delta_1| \leq 1, \quad |\delta_2| \leq \frac{1}{2} + \frac{1}{e^2} = 0,635\ldots$$

and are unknown for $n \geq 3$.

So it is natural to ask the sharp estimates of $|\delta_n|$ $(n \in \mathbb{N})$ for functions belonging to the subclasses of univalent function class $\mathcal{S}$. One of the main purpose of this paper is to determine the sharp upper bound for $|\delta_n|$ $(n \in \mathbb{N})$ of the function f belonging to the class $\mathcal{S}_p(\alpha)$. Some recent works on logarithmic coefficients can be found in [2, 4, 17].

On the other hand, one of the important tools in the theory of univalent functions are the Hankel determinants, which are used, for example, in showing that a function of bounded characteristic in $\mathbb{U}$, that is, a function that is a ratio of two bounded analytic functions, with its Laurent series around the origin having integral coefficients, is rational [8].

For integers $n, q \in \mathbb{N}$, Noonan and Thomas [19] defined the qth Hankel determinant $H_{q,n}(f)$ of $f \in \mathcal{A}$ of the form (1.1) by

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (a_1 = 1).$$

Note that

$$H_{2,1}(f) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} \qquad \text{and} \qquad H_{2,2}(f) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix},$$

where the Hankel determinants $H_{2,1}(f) = a_3 - a_2^2$ and $H_{2,2}(f) = a_2a_4 - a_3^2$ are well known as Fekete-Szegö and the second Hankel determinant functionals, respectively. Furthermore, Fekete and Szegö [12] introduced the generalized functional $a_3 - \lambda a_2^2$, where λ is some real number. Problems in this field have also been argued by several authors (see, for example, [1, 6, 7, 10, 13, 18, 21]).

Very recently, Kowalczyk and Lecko [14] introduced the Hankel determinant $H_{q,n}\left(\frac{F_f}{2}\right)$, which entries are logarithmic coefficients of f, that is,

$$H_{q,n}\left(\frac{F_f}{2}\right) = \begin{vmatrix} \delta_n & \delta_{n+1} & \dots & \delta_{n+q-1} \\ \delta_{n+1} & \delta_{n+2} & \dots & \delta_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{n+q-1} & \delta_{n+q} & \dots & \delta_{n+2q-2} \end{vmatrix}.$$

The main purpose of this paper is to investigate the upper bound of

$$H_{2,1}\left(\frac{F_f}{2}\right) = \delta_1\delta_3 - \delta_2^2$$

and of logarithmic coefficients δ_n for functions belonging to the classes $\mathcal{S}_p(\alpha)$ and $\mathcal{UCV}(\alpha)$.

2. Preliminary lemmas

Throughout this paper, we assume that

$$p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots \quad (z \in \mathbb{U}). \tag{2.1}$$

To prove our main results, we need the following lemmas.

Lemma 2.1 ([20]). *Let $p \in \mathcal{P}$ be given by (2.1). Then*

$$|c_n| \leq 2 \quad (n \in \mathbb{N}).$$

Lemma 2.2 ([21]). *Let $p \in \mathcal{P}$ be given by (2.1). Then for any complex number ν*

$$\left|c_2 - \nu c_1^2\right| \leq 2\max\{1, |2\nu - 1|\},$$

and the result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \quad \text{and} \quad p(z) = \frac{1+z}{1-z}.$$

Lemma 2.3 ([16]). *Let $p \in \mathcal{P}$ be given by (2.1). Then*

$$\left|c_2 - \nu c_1^2\right| \leq \begin{cases} -4\nu + 2, & \nu \leq 0, \\ 2, & 0 \leq \nu \leq 1, \\ 4\nu - 2, & \nu \geq 1. \end{cases}$$

When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p(z)$ is $\frac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p(z)$ is $\frac{1+z^2}{1-z^2}$ or one of its rotations. If $\nu = 0$, then the equality holds if and only if

$$p(z) = \left(\frac{1}{2} + \frac{1}{2}\eta\right)\frac{1+z}{1-z} + \left(\frac{1}{2} - \frac{1}{2}\eta\right)\frac{1-z}{1+z} \quad (0 \leq \eta \leq 1)$$

or one of its rotations. If $\nu = 1$*, then the equality holds if and only if* $p(z)$ *is the reciprocal of one of the functions such that the equality holds in the case when* $\nu = 0$*.*

Although the above upper bound is sharp, in the case when $0 < \nu < 1$*, it can be further improved as follows:*

$$\left|c_2 - \nu c_1^2\right| + \nu \left|c_1\right|^2 \leq 2 \qquad \left(0 < \nu \leq \frac{1}{2}\right)$$

and

$$\left|c_2 - \nu c_1^2\right| + (1-\nu) \left|c_1\right|^2 \leq 2 \qquad \left(\frac{1}{2} < \nu \leq 1\right).$$

Lemma 2.4 ([9])**.** *If* $p \in \mathcal{P}$ *is of the form* (2.1) *with* $c_1 \geq 0$*, then*

$$\begin{cases} c_1 = 2\zeta_1, \\ c_2 = 2\zeta_1^2 + 2\left(1-\zeta_1^2\right)\zeta_2, \\ c_3 = 2\zeta_1^3 + 4\left(1-\zeta_1^2\right)\zeta_1\zeta_2 - 2\left(1-\zeta_1^2\right)\zeta_1\zeta_2^2 + 2\left(1-\zeta_1^2\right)\left(1-\left|\zeta_2\right|^2\right)\zeta_3 \end{cases} \tag{2.2}$$

for some $\zeta_1 \in [0,1]$ *and* $\zeta_2, \zeta_3 \in \overline{\mathbb{U}} = \{z \in \mathbb{C} : |z| \leq 1\}$*.*

For $\zeta_1 \in \mathbb{U}$ *and* $\zeta_2 \in \partial\mathbb{U} = \{z \in \mathbb{C} : |z| = 1\}$*, there is a unique function* $p \in \mathcal{P}$ *with* c_1 *and* c_2 *as in* (2.2)*, namely,*

$$p(z) = \frac{1 + \left(\overline{\zeta_1}\zeta_2 + \zeta_1\right) z + \zeta_2 z^2}{1 + \left(\overline{\zeta_1}\zeta_2 - \zeta_1\right) z - \zeta_2 z^2} \qquad (z \in \mathbb{U}).$$

Lemma 2.5 ([22])**.** *Let the function* $\mathfrak{h}$ *given by*

$$\mathfrak{h}(z) = 1 + \sum_{k=1}^{\infty} \mathfrak{h}_k z^k \qquad (z \in \mathbb{U})$$

be subordinate to the function $\mathfrak{H}$ *given by*

$$\mathfrak{H}(z) = 1 + \sum_{k=1}^{\infty} \mathfrak{H}_k z^k \qquad (z \in \mathbb{U}).$$

If $\mathfrak{H}(z)$ *is univalent in* $\mathbb{U}$ *and* $\mathfrak{H}(\mathbb{U})$ *is convex, then*

$$|\mathfrak{h}_k| \leq |\mathfrak{H}_1| \qquad (k \in \mathbb{N}).$$

Lemma 2.6 ([10])**.** *Given real numbers* A, B, C*, let*

$$Y(A,B,C) := \max_{z \in \overline{\mathbb{U}}} \left(\left|A + Bz + Cz^2\right| + 1 - |z|^2\right).$$

I. If $AC \geq 0$*, then*

$$Y(A,B,C) = \begin{cases} |A| + |B| + |C|, & |B| \geq 2(1-|C|), \\ 1 + |A| + \frac{B^2}{4(1-|C|)}, & |B| < 2(1-|C|). \end{cases}$$

II. If $AC < 0$*, then*

$$Y(A,B,C) = \begin{cases} 1 - |A| + \frac{B^2}{4(1-|C|)}, & -4AC\left(C^{-2}-1\right) \leq B^2 \ \wedge \ |B| < 2\left(1-|C|\right), \\ 1 + |A| + \frac{B^2}{4(1+|C|)}, & B^2 < \min\left\{ 4\left(1+|C|\right)^2, -4AC\left(C^{-2}-1\right) \right\}, \\ R(A,B,C), & \textit{otherwise}, \end{cases}$$

where

$$R(A,B,C) = \begin{cases} |A| + |B| - |C|, & |C|\left(|B| + 4|A|\right) \leq |AB|, \\ -|A| + |B| + |C|, & |AB| \leq |C|\left(|B| - 4|A|\right), \\ \left(|A| + |C|\right)\sqrt{1 - \frac{B^2}{4AC}} & \textit{otherwise}. \end{cases}$$

Lemma 2.7 ([3])**.** *Let* φ *be an analytic univalent function in the unit disk* $\mathbb{U}$ *satisfying* $\varphi(0) = 1$ *such that it has series expansion of the form*

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots, \quad B_1 \neq 0.$$

If φ *is convex and the function* f *given by* (1.1) *satisfies the subordination*

$$\frac{zf'(z)}{f(z)} \prec \varphi(z) \qquad (z \in \mathbb{U}),$$

then the logarithmic coefficients δ_n *of* f *satisfy the inequality*

$$|\delta_n| \leq \frac{|B_1|}{2n} \qquad (n \in \mathbb{N}).$$

3. The class $\mathcal{S}_p(\alpha)$

3.1. The logarithmic coefficients.

Theorem 3.1. *Let* $f \in \mathcal{S}_p(\alpha)$ $(0 \leq \alpha < 1)$ *be given by* (1.1) *and let the coefficients of* $\log\left(f(z)/z\right)$ *be given by* (1.7)*. Then*

$$|\delta_n| \leq \frac{8(1-\alpha)}{n\pi^2} \qquad (n \in \mathbb{N}). \tag{3.1}$$

For each $n \in \mathbb{N}$*, there exist a function* f_n *given by*

$$\frac{zf_n'(z)}{f_n(z)} = q_\alpha(z^n) \quad (n \in \mathbb{N})$$

such that the each equality in (3.1) *is sharp.*

Proof. The proof is easily obtained from Lemma 2.7. □

Corollary 3.2. *Let* $f \in \mathcal{S}_p$ *be given by* (1.1) *and let the coefficients of* $\log\left(f(z)/z\right)$ *be given by* (1.7)*. Then*

$$|\delta_n| \leq \frac{4}{n\pi^2} \qquad (n \in \mathbb{N}).$$

The result is sharp.

Theorem 3.3. *Let $f \in \mathcal{S}_p(\alpha)$ $(0 \leq \alpha < 1)$ be given by (1.1) and let the coefficients of* $\log(f(z)/z)$ *be given by* (1.7)*. Then*

$$\left|\delta_2 - \mu\delta_1^2\right| \leq \begin{cases} \frac{8(1-\alpha)}{\pi^2}\left(\frac{1}{3} - \frac{8(1-\alpha)}{\pi^2}\mu\right), & \mu \leq -\frac{\pi^2}{48(1-\alpha)}, \\ \frac{4(1-\alpha)}{\pi^2}, & -\frac{\pi^2}{48(1-\alpha)} \leq \mu \leq \frac{5\pi^2}{48(1-\alpha)}, \\ -\frac{8(1-\alpha)}{\pi^2}\left(\frac{1}{3} - \frac{8(1-\alpha)}{\pi^2}\mu\right), & \mu \geq \frac{5\pi^2}{48(1-\alpha)}. \end{cases}$$

If $-\frac{\pi^2}{48(1-\alpha)} \leq \mu \leq \frac{\pi^2}{24(1-\alpha)}$, *then*

$$\left|\delta_2 - \mu\delta_1^2\right| + \left(\mu + \frac{\pi^2}{48(1-\alpha)}\right)|\delta_1|^2 \leq \frac{4(1-\alpha)}{\pi^2}.$$

Furthermore, if $\frac{\pi^2}{24(1-\alpha)} \leq \mu \leq \frac{5\pi^2}{48(1-\alpha)}$, *then*

$$\left|\delta_2 - \mu\delta_1^2\right| + \left(\frac{5\pi^2}{48(1-\alpha)} - \mu\right)|\delta_1|^2 \leq \frac{4(1-\alpha)}{\pi^2}.$$

Each of these results is sharp.

Proof. Let $f \in \mathcal{S}_p(\alpha)$. By the subordination principle and Lemma 1.4, there exists the Schwarz's function $u(z)$ such that

$$F(z) := \frac{zf'(z)}{f(z)} = q_\alpha(u(z)) \qquad (z \in \mathbb{U}). \tag{3.2}$$

If

$$F(z) = 1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots,$$

then the first equality in (3.2) implies that

$$a_2 = b_1, \qquad a_3 = \frac{1}{2}\left(b_2 + b_1^2\right), \qquad a_4 = \frac{1}{3}\left(b_3 + \frac{3}{2}b_1 b_2 + \frac{1}{2}b_1^3\right). \tag{3.3}$$

Since q_α is univalent in the open unit disk $\mathbb{U}$, by (3.2), the function

$$p(z) := \frac{1+u(z)}{1-u(z)} = \frac{1+q_\alpha^{-1}(F(z))}{1-q_\alpha^{-1}(F(z))} = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \tag{3.4}$$

belongs to the class $\mathcal{P}$. Solving $u(z)$ in terms of $p(z)$ in (3.4), we obtain

$$u(z) = \frac{1}{2}\left[c_1 z + \left(c_2 - \frac{c_1^2}{2}\right)z^2 + \left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right)z^3 + \cdots\right]. \tag{3.5}$$

In view of (3.2), using (3.5) in (1.5), we find

$$\begin{aligned} &1 + b_1 z + b_2 z^2 + b_3 z^3 + \cdots \\ &\quad = 1 + \frac{1}{2}D_1 c_1 z + \left\{\frac{1}{2}D_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{4}D_2 c_1^2\right\}z^2 \\ &\qquad + \left\{\frac{1}{2}D_1\left(c_3 - c_1 c_2 + \frac{c_1^3}{4}\right) + \frac{1}{2}D_2 c_1\left(c_2 - \frac{c_1^2}{2}\right) + \frac{1}{8}D_3 c_1^3\right\}z^3 + \cdots. \end{aligned}$$

Equating the coefficients in the above equalities and considering (1.6), we have

$$b_1 = \frac{8(1-\alpha)}{\pi^2}c_1, \qquad b_2 = \frac{8(1-\alpha)}{\pi^2}\left(c_2 - \frac{c_1^2}{6}\right), \tag{3.6}$$

and

$$b_3 = \frac{8(1-\alpha)}{\pi^2}\left(c_3 - \frac{1}{3}c_1c_2 + \frac{2}{45}c_1^3\right). \tag{3.7}$$

Using (3.3) in (3.6) and (3.7), we get

$$a_2 = \frac{8(1-\alpha)}{\pi^2}c_1, \tag{3.8}$$

$$a_3 = \frac{8(1-\alpha)}{2\pi^2}\left[c_2 - \left(\frac{1}{6} - \frac{8(1-\alpha)}{\pi^2}\right)c_1^2\right], \tag{3.9}$$

$$\begin{aligned} a_4 = {} & \frac{8(1-\alpha)}{3\pi^2}\left[c_3 - \left(\frac{1}{3} - \frac{12(1-\alpha)}{\pi^2}\right)c_1c_2\right. \\ & \left. + \left(\frac{2}{45} - \frac{2(1-\alpha)}{\pi^2} + \frac{32(1-\alpha)^2}{\pi^4}\right)c_1^3\right]. \end{aligned} \tag{3.10}$$

For δ_1, from (1.8) and (3.8), we have

$$\delta_1 = \frac{4(1-\alpha)}{\pi^2}c_1, \tag{3.11}$$

and for δ_2, substituting for a_2 and a_3 from (3.8) and (3.9) in (1.9), we obtain

$$\delta_2 = \frac{2(1-\alpha)}{\pi^2}\left(c_2 - \frac{1}{6}c_1^2\right). \tag{3.12}$$

Furthermore, from (1.8) and (3.8) – (3.10), we get

$$\delta_3 = \frac{4(1-\alpha)}{3\pi^2}\left(c_3 - \frac{1}{3}c_1c_2 + \frac{2}{45}c_1^3\right). \tag{3.13}$$

Then from (3.11) and (3.12), we get

$$\left|\delta_2 - \mu\delta_1^2\right| = \frac{2(1-\alpha)}{\pi^2}\left|c_2 - \nu c_1^2\right|, \qquad \nu = \frac{1}{6} + \frac{8(1-\alpha)}{\pi^2}\mu.$$

The assertion of Theorem 3.3 now follows by an application of Lemma 2.3.

To show that the bounds asserted by Theorem 3.3 are sharp, we define the following functions:

$$K_n(z) \qquad (n = 2, 3, \dots),$$

by

$$K_n(0) = 0 = K_n'(0) - 1,$$

and

$$\frac{zK_n'(z)}{K_n(z)} = q_\alpha\left(z^{n-1}\right),$$

and the functions $F_\eta(z)$ and $G_\eta(z)$ $(0 \leq \eta \leq 1)$ by

$$F_\eta(0) = 0 = F_\eta'(0) - 1 \quad \text{and} \quad G_\eta(0) = 0 = G_\eta'(0) - 1,$$

$$\frac{zF_\eta'(z)}{F_\eta(z)} = q_\alpha\left(\frac{z(z+\eta)}{1+\eta z}\right),$$

and

$$\frac{zG_\eta'(z)}{G_\eta(z)} = q_\alpha\left(-\frac{z(z+\eta)}{1+\eta z}\right),$$

respectively. Then, clearly, the functions $K_n, F_\eta, G_\eta \in \mathcal{S}_p(\alpha)$. We also write $K = K_2$. If $\mu < -\frac{\pi^2}{48(1-\alpha)}$ or $\mu > \frac{5\pi^2}{48(1-\alpha)}$, then the equality of Theorem 3.3 holds if and only if f is K or one of its rotations. When $-\frac{\pi^2}{48(1-\alpha)} < \mu < \frac{5\pi^2}{48(1-\alpha)}$, then the equality holds if and only if f is K_3 or one of its rotations. If $\mu = -\frac{\pi^2}{48(1-\alpha)}$, then the equality holds if and only if f is F_η or one of its rotations. If $\mu = \frac{5\pi^2}{48(1-\alpha)}$, then the equality holds if and only if f is G_η or one of its rotations. □

Corollary 3.4. *Let $f \in \mathcal{S}_p$ be given by (1.1) and let the coefficients of* $\log(f(z)/z)$ *be given by (1.7). Then*

$$\left|\delta_2 - \mu\delta_1^2\right| \le \begin{cases} \frac{4}{\pi^2}\left(\frac{1}{3} - \frac{4}{\pi^2}\mu\right), & \mu \le -\frac{\pi^2}{24}, \\ \frac{2}{\pi^2}, & -\frac{\pi^2}{24} \le \mu \le \frac{5\pi^2}{24} \\ -\frac{4}{\pi^2}\left(\frac{1}{3} - \frac{4}{\pi^2}\mu\right), & \mu \ge \frac{5\pi^2}{24}. \end{cases}$$

If $-\frac{\pi^2}{24} \le \mu \le \frac{\pi^2}{12}$, then

$$\left|\delta_2 - \mu\delta_1^2\right| + \left(\mu + \frac{\pi^2}{24}\right)|\delta_1|^2 \le \frac{2}{\pi^2}.$$

Furthermore, if $\frac{\pi^2}{12} \le \mu \le \frac{5\pi^2}{24}$, then

$$\left|\delta_2 - \mu\delta_1^2\right| + \left(\frac{5\pi^2}{24} - \mu\right)|\delta_1|^2 \le \frac{2}{\pi^2}.$$

Each of these results is sharp.

Theorem 3.5. *Let $f \in \mathcal{S}_p(\alpha)$ be given by (1.1) and let the coefficients of* $\log(f(z)/z)$ *be given by (1.7). Then for $\mu \in \mathbb{C}$ and*

$$\chi(\mu) = \frac{1}{3} + \frac{16(1-\alpha)}{\pi^2}\mu,$$

we have

$$\left|\delta_2 - \mu\delta_1^2\right| \le \begin{cases} \frac{8(1-\alpha)}{3\pi^4}\left|24(1-\alpha)\mu - \pi^2\right|, & |\chi(\mu) - 1| \ge 1, \\ \frac{4(1-\alpha)}{\pi^2}, & |\chi(\mu) - 1| \le 1. \end{cases}$$

Proof. From (1.8), (3.8) and (3.12), we get

$$\left|\delta_2 - \mu\delta_1^2\right| = \frac{2(1-\alpha)}{\pi^2}\left|c_2 - \nu c_1^2\right|, \qquad \nu = \frac{1}{6} + \frac{8(1-\alpha)}{\pi^2}\mu$$

for any $\mu \in \mathbb{C}$. The desired result is obtained from Lemma 2.2. □

Corollary 3.6. *Let $f \in \mathcal{S}_p$ be given by* (1.1) *and let the coefficients of* $\log(f(z)/z)$ *be given by* (1.7). *Then for $\mu \in \mathbb{C}$ and*

$$\chi(\mu) = \frac{1}{3} + \frac{8}{\pi^2}\mu,$$

we have

$$\left|\delta_2 - \mu\delta_1^2\right| \leq \begin{cases} \frac{4}{3\pi^4}\left|12\mu - \pi^2\right|, & |\chi(\mu) - 1| \geq 1, \\ \frac{2}{\pi^2}, & |\chi(\mu) - 1| \leq 1. \end{cases}$$

3.2. **Second Hankel determinant.**

Theorem 3.7. *Let $f \in \mathcal{S}_p(\alpha)$ $(0 \leq \alpha < 1)$ be given by* (1.1) *and let the coefficients of* $\log(f(z)/z)$ *be given by* (1.7). *Then*

$$\left|\delta_1\delta_3 - \delta_2^2\right| \leq \frac{16(1-\alpha)^2}{\pi^4}.$$

The inequality is sharp.

Proof. Suppose that $f \in \mathcal{S}_p(\alpha)$ is given by (1.1). By using (3.11) – (3.13) and Lemma 2.4, we obtain

$$\begin{aligned} \delta_1\delta_3 - \delta_2^2 &= \frac{4(1-\alpha)^2}{\pi^4}\left[\frac{17}{540}c_1^4 - \frac{1}{9}c_1^2c_2 + \frac{4}{3}c_1c_3 - c_2^2\right] \\ &= \frac{16(1-\alpha)^2}{3\pi^4}\left[\frac{32}{45}\zeta_1^4 + \frac{4}{3}\left(1-\zeta_1^2\right)\zeta_1^2\zeta_2 - \left(1-\zeta_1^2\right)\left(\zeta_1^2+3\right)\zeta_2^2\right. \\ &\quad \left. +4\left(1-\zeta_1^2\right)\left(1-|\zeta_2|^2\right)\zeta_1\zeta_3\right]. \end{aligned} \tag{3.14}$$

(a) Firstly suppose that $\zeta_1 = 1$. Then by (3.14), we have

$$\left|\delta_1\delta_3 - \delta_2^2\right| = \frac{512}{135\pi^4}(1-\alpha)^2.$$

(b) Now, suppose that $\zeta_1 = 0$. Then by (3.14),

$$\left|\delta_1\delta_3 - \delta_2^2\right| = \frac{16(1-\alpha)^2}{\pi^4}|\zeta_2|^2 \leq \frac{16(1-\alpha)^2}{\pi^4}.$$

(c) Finally, suppose that $\zeta_1 \in (0,1)$. By the fact that $|\zeta_3| \leq 1$, from (3.14), we get

$$\begin{aligned} \left|\delta_1\delta_3 - \delta_2^2\right| &\leq \frac{64(1-\alpha)^2}{3\pi^4}\zeta_1\left(1-\zeta_1^2\right) \\ &\quad \times\left[\left|\frac{8\zeta_1^3}{45\left(1-\zeta_1^2\right)} + \frac{\zeta_1\zeta_2}{3} - \frac{\left(\zeta_1^2+3\right)\zeta_2^2}{4\zeta_1}\right| + 1 - |\zeta_2|^2\right] \\ &= \frac{64(1-\alpha)^2}{3\pi^4}\zeta_1\left(1-\zeta_1^2\right)\left[\left|A + B\zeta_2 + C\zeta_2^2\right| + 1 - |\zeta_2|^2\right], \end{aligned}$$

where

$$A := \frac{8\zeta_1^3}{45\left(1-\zeta_1^2\right)}, \quad B := \frac{\zeta_1}{3}, \quad C := -\frac{\zeta_1^2+3}{4\zeta_1}.$$

Since $AC < 0$, we apply Lemma 2.6 only for the case II.

(c.1) The inequality

$$-4AC\left(\frac{1}{C^2}-1\right)-B^2=\frac{8\zeta_1^2\left(\zeta_1^2+3\right)}{45\left(1-\zeta_1^2\right)}\left(\frac{16\zeta_1^2}{\left(\zeta_1^2+3\right)^2}-1\right)-\frac{\zeta_1^2}{9}\leq 0$$

is equivalent to $-\zeta_1^4+30\zeta_1^2-29\leq 0$, which evidently holds for $\zeta_1\in(0,1)$. Moreover, the inequality $|B|<2\left(1-|C|\right)$ is equivalent to $\frac{5}{3}\zeta_1^2-4\zeta_1+3<0$, which is false for $\zeta_1\in(0,1)$.

(c.2) Since

$$4\left(1+|C|\right)^2=\frac{\left(\zeta_1+1\right)^2\left(\zeta_1+3\right)^2}{4\zeta_1^2}>0$$

and

$$-4AC\left(\frac{1}{C^2}-1\right)=\frac{8\zeta_1^2\left(\zeta_1^2-9\right)}{45\left(\zeta_1^2+3\right)}<0,$$

we see that the inequality

$$\frac{\zeta_1^2}{9}<\min\left\{\frac{\left(\zeta_1+1\right)^2\left(\zeta_1+3\right)^2}{4\zeta_1^2},\frac{8\zeta_1^2\left(\zeta_1^2-9\right)}{45\left(\zeta_1^2+3\right)}\right\}=\frac{8\zeta_1^2\left(\zeta_1^2-9\right)}{45\left(\zeta_1^2+3\right)}$$

is false for $\zeta_1\in(0,1)$.

(c.3) The inequality

$$|C|\left(|B|+4|A|\right)-|AB|=\frac{\left(\zeta_1^2+3\right)}{12}\left(1+\frac{32\zeta_1^2}{15\left(1-\zeta_1^2\right)}\right)-\frac{8\zeta_1^4}{135\left(1-\zeta_1^2\right)}\leq 0$$

is equivalent to

$$19\zeta_1^4+198\zeta_1^2+135\leq 0,$$

which is false for $\zeta_1\in(0,1)$.

(c.4) We get

$$|AB|-|C|\left(|B|-4|A|\right)=\frac{173\zeta_1^4+378\zeta_1^2-135}{540\left(1-\zeta_1^2\right)}:=\frac{173s^2+378s-135}{540\left(1-s\right)},$$

where $s=\zeta_1^2\in(0,1)$. The equation $173s^2+378s-135=0$ has a positive unique root such that

$$0<s_1=\frac{-189+6\sqrt{1641}}{173}<1.$$

In other words, for $\zeta_1^*=\sqrt{s_1}$, we have $|AB|-|C|\left(|B|-4|A|\right)=0$. Furthermore, $|AB|\leq|C|\left(|B|-4|A|\right)$ when $\zeta_1\in(0,\zeta_1^*]$ and $|AB|\geq|C|\left(|B|-4|A|\right)$ when $\zeta_1\in[\zeta_1^*,1)$.

- For $\zeta_1\in(0,\zeta_1^*]$, we obtain

$$\begin{aligned}\left|\delta_1\delta_3-\delta_2^2\right| &\leq \frac{64\left(1-\alpha\right)^2}{3\pi^4}\zeta_1\left(1-\zeta_1^2\right)\left(-|A|+|B|+|C|\right)\\ &= \frac{16\left(1-\alpha\right)^2}{135\pi^4}\left[-137\zeta_1^4-30\zeta_1^2+135\right]\\ &=: \chi\left(\zeta_1\right).\end{aligned}$$

Since

$$\chi'\left(\zeta_1\right)=-\frac{64\left(1-\alpha\right)^2}{135\pi^4}\zeta_1\left[137\zeta_1^2+15\right]<0$$

for $\zeta_1 \in (0, \zeta_1^*]$, χ is a decreasing function on $(0, \zeta_1^*]$. This implies that

$$\left|\delta_1\delta_3 - \delta_2^2\right| \leq \chi(0) = \frac{16(1-\alpha)^2}{\pi^4}.$$

- For $\zeta_1 \in [\zeta_1^*, 1)$, we obtain

$$\begin{aligned}
\left|\delta_1\delta_3 - \delta_2^2\right| &\leq \frac{64(1-\alpha)^2}{3\pi^4}\zeta_1\left(1-\zeta_1^2\right)(|A|+|C|)\sqrt{1-\frac{B^2}{4AC}} \\
&= \frac{16(1-\alpha)^2}{135\pi^4}\left[-13\zeta_1^4 - 90\zeta_1^2 + 135\right]\sqrt{\frac{3\zeta_1^2+29}{8(\zeta_1^2+3)}} \\
&=: \psi(\zeta_1).
\end{aligned}$$

Since

$$\begin{aligned}
\psi'(\zeta_1) = &-\frac{16(1-\alpha)^2}{135\pi^4}\zeta_1 \\
&\times\left[2\left(13\zeta_1^2+45\right)\sqrt{\frac{3\zeta_1^2+29}{2(\zeta_1^2+3)}} + 5\frac{-13\zeta_1^4-90\zeta_1^2+135}{(\zeta_1^2+3)^2}\sqrt{\frac{2(\zeta_1^2+3)}{3\zeta_1^2+29}}\right] < 0
\end{aligned}$$

for $\zeta_1 \in [\zeta_1^*, 1)$, ψ is a decreasing function on $[\zeta_1^*, 1)$. This implies that

$$\left|\delta_1\delta_3 - \delta_2^2\right| \leq \psi(\zeta_1) \leq \psi(\zeta_1^*) = \chi(\zeta_1^*) \leq \chi(0) = \frac{16(1-\alpha)^2}{\pi^4}.$$

Summarizing parts (a)-(c), it follows the desired inequality. Equality holds for the function $f \in \mathcal{A}$ given by

$$\frac{zf'(z)}{f(z)} = q_\alpha(z^2) \qquad (z \in \mathbb{U})$$

for which $a_2 = a_4 = 0$ and $a_3 = \frac{8(1-\alpha)}{\pi^2}$. □

Corollary 3.8. *Let $f \in \mathcal{S}_p$ be given by (1.1) and let the coefficients of* $\log(f(z)/z)$ *be given by (1.7). Then we have*

$$\left|\delta_1\delta_3 - \delta_2^2\right| \leq \frac{4}{\pi^4}.$$

The inequality is sharp.

3.3. **The coefficients of the inverse function.** Since univalent functions are one-to-one, they are invertible and the inverse functions need not to be defined on the entire unit disk $\mathbb{U}$. In fact, the Koebe one-quarter theorem [11] ensures that the image of $\mathbb{U}$ under every univalent function $f \in \mathcal{S}$ contains a disk of radius $1/4$. Thus every function $f \in \mathcal{A}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f\left(f^{-1}(w)\right) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right).$$

In fact, for a function $f \in \mathcal{A}$ given by (1.1) the inverse function f^{-1} is given by

$$\begin{aligned} f^{-1}(w) &= w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots \\ &=: w + \sum_{n=2}^{\infty} A_n w^n. \end{aligned} \tag{3.15}$$

Theorem 3.9. *Let $f \in \mathcal{S}_p(\alpha)$ $(0 \le \alpha < 1)$ be given by (1.1), and let f^{-1} be the inverse function of f defined by (3.15). Then*

$$|A_2| \le \frac{16(1-\alpha)}{\pi^2},$$

$$|A_3| \le \begin{cases} \frac{16(1-\alpha)}{3\pi^4}\left[72(1-\alpha) - \pi^2\right], & 0 \le \alpha \le 1 - \frac{5\pi^2}{144}, \\ \frac{8(1-\alpha)}{\pi^2}, & 1 - \frac{5\pi^2}{144} \le \alpha < 1, \end{cases}$$

and for $\lambda \in \mathbb{C}$

$$\left|A_3 - \lambda A_2^2\right| \le \begin{cases} \frac{16(1-\alpha)}{3\pi^4}\left|48(1-\alpha)\lambda - 72(1-\alpha) + \pi^2\right|, & |h(\lambda) - 1| \ge 1, \\ \frac{8(1-\alpha)}{\pi^2}, & |h(\lambda) - 1| \le 1, \end{cases}$$

where

$$h(\lambda) = \frac{1}{3} + \frac{48(1-\alpha)}{\pi^2} - \lambda \frac{32(1-\alpha)}{\pi^2}.$$

Proof. Let the function $f \in \mathcal{A}$ given by (1.1) be in the class $\mathcal{S}_p(\alpha)$, and let f^{-1} be the inverse function of f defined by (3.15). Then using (3.8)-(3.12), we obtain

$$A_2 = -a_2 = -\frac{8(1-\alpha)}{\pi^2} c_1,$$

$$\begin{aligned} A_3 &= 2a_2^2 - a_3 = \left(\frac{96(1-\alpha)^2}{\pi^4} + \frac{2(1-\alpha)}{3\pi^2}\right) c_1^2 - \frac{4(1-\alpha)}{\pi^2} c_2 \\ &= -\frac{4(1-\alpha)}{\pi^2}\left[c_2 - \left(\frac{1}{6} + \frac{24(1-\alpha)}{\pi^2}\right) c_1^2\right], \end{aligned}$$

and

$$A_3 - \lambda A_2^2 = -\frac{4(1-\alpha)}{\pi^2}\left[c_2 - \left(\frac{1}{6} + \frac{24(1-\alpha)}{\pi^2} + \lambda \frac{16(1-\alpha)}{\pi^2}\right) c_1^2\right].$$

The inequality for $|A_2|$ is obtained by the means of Lemma 2.1. An application of Lemma 2.3 gives the inequality for $|A_3|$. On the other hand, we find the upper bound on $|A_3 - \lambda A_2^2|$ from Lemma 2.2. $\square$

Corollary 3.10. *Let $f \in \mathcal{S}_p$ be given by (1.1), and let f^{-1} be the inverse function of f defined by (3.15). Then*

$$|A_2| \le \frac{8}{\pi^2}, \qquad |A_3| \le \frac{8}{3\pi^4}\left(36 - \pi^2\right),$$

and for $\lambda \in \mathbb{C}$,

$$\left|A_3-\lambda A_2^2\right| \le \begin{cases} \frac{8}{3\pi^4}\left|24\lambda-36+\pi^2\right|, & \left|h(\lambda)-1\right| \ge 1, \\ \frac{4}{\pi^2}, & \left|h(\lambda)-1\right| \le 1, \end{cases}$$

where

$$h(\lambda)=\frac{1}{3}+\frac{24}{\pi^2}-\lambda\frac{16}{\pi^2}.$$

4. The class $\mathcal{UCV}(\alpha)$

4.1. The logarithmic coefficients.

Theorem 4.1. *Let* $f \in \mathcal{UCV}(\alpha)$ $(0 \le \alpha < 1)$ *be given by* (1.1) *and let the coefficients of* $\log(f(z)/z)$ *be given by* (1.7). *Then*

$$\begin{aligned} |\delta_1| &\le \frac{4(1-\alpha)}{\pi^2}, \\ |\delta_2| &\le \begin{cases} \frac{8(1-\alpha)}{3\pi^2}\left(\frac{1}{3}+\frac{2(1-\alpha)}{\pi^2}\right), & 0 \le \alpha \le 1-\frac{\pi^2}{12}, \\ \frac{4(1-\alpha)}{3\pi^2}, & 1-\frac{\pi^2}{12} \le \alpha < 1, \end{cases} \\ |\delta_3| &\le \frac{2(1-\alpha)}{3\pi^2}+\frac{16(1-\alpha)^2}{3\pi^4}. \end{aligned}$$

The bounds for $|\delta_1|$ *and* $|\delta_2|$ *are sharp.*

Proof. If $f \in \mathcal{UCV}(\alpha)$, then from (1.2), we know that $zf' \in \mathcal{S}_p(\alpha)$. Define the function g by

$$g(z)=zf'(z)=z+\sum_{n=2}^{\infty} d_n z^n \qquad (z \in \mathbb{U}), \tag{4.1}$$

and consider the logarithmic coefficients γ_n $(n \in \mathbb{N})$ defined by

$$F_g(z):=\log\frac{g(z)}{z}=2\sum_{n=1}^{\infty}\gamma_n z^n \qquad (z \in \mathbb{U}). \tag{4.2}$$

Therefore

$$\gamma_1=\frac{1}{2}d_2, \tag{4.3}$$

$$\gamma_2=\frac{1}{2}\left(d_3-\frac{1}{2}d_2^2\right), \tag{4.4}$$

$$\gamma_3=\frac{1}{2}\left(d_4-d_2d_3+\frac{1}{3}d_2^3\right). \tag{4.5}$$

By equating the coefficients of z^n reciprocally in (4.1), we get $na_n=d_n$ for all $n \in \mathbb{N}$. On the other hand, since $g=zf' \in \mathcal{S}_p(\alpha)$, considering the logarithmic coefficients γ_n given by (4.2) and using (1.8) – (1.10), the logarithmic coefficients of the function $f \in \mathcal{UCV}(\alpha)$ are obtained equal to

$$\begin{cases} \delta_1 = \frac{1}{2}\gamma_1, \\ \delta_2 = \frac{1}{3}\left(\gamma_2 + \frac{1}{4}\gamma_1^2\right), \\ \delta_3 = \frac{1}{4}\left(\gamma_3 + \frac{2}{3}\gamma_1\gamma_2\right). \end{cases} \tag{4.6}$$

By letting $n = 1$ in Theorem 3.1, we obtain

$$|\delta_1| = \frac{1}{2}|\gamma_1| \leq \frac{4(1-\alpha)}{\pi^2}.$$

Next, the upper bound of $|\delta_2|$ is obtained by Theorem 3.3 for $\mu = -1/4$. So we get

$$|\delta_2| = \frac{1}{3}\left|\gamma_2 + \frac{1}{4}\gamma_1^2\right| \leq \begin{cases} \frac{8(1-\alpha)}{3\pi^2}\left(\frac{1}{3} + \frac{2(1-\alpha)}{\pi^2}\right), & 0 \leq \alpha \leq 1 - \frac{\pi^2}{12}, \\ \frac{4(1-\alpha)}{3\pi^2}, & 1 - \frac{\pi^2}{12} \leq \alpha < 1. \end{cases}$$

Finally, for $|\delta_3|$,

$$\begin{aligned} |\delta_3| &= \frac{1}{4}\left|\gamma_3 + \frac{2}{3}\gamma_1\gamma_2\right| \\ &\leq \frac{1}{4}\left[|\gamma_3| + \frac{2}{3}|\gamma_1||\gamma_2|\right] \\ &\leq \frac{2(1-\alpha)}{3\pi^2} + \frac{16(1-\alpha)^2}{3\pi^4}. \end{aligned}$$

□

Corollary 4.2. *Let $f \in \mathcal{UCV}$ be given by (1.1) and let the coefficients of $\log(f(z)/z)$ be given by (1.7). Then*

$$|\delta_1| \leq \frac{2}{\pi^2}, \qquad |\delta_2| \leq \frac{2}{3\pi^2}, \qquad |\delta_3| \leq \frac{1}{3\pi^2} + \frac{4}{3\pi^4}.$$

The bounds for $|\delta_1|$ and $|\delta_2|$ are sharp.

Theorem 4.3. *Let $f \in \mathcal{UCV}(\alpha)$ $(0 \leq \alpha < 1)$ be given by (1.1) and let the coefficients of $\log(f(z)/z)$ be given by (1.7). Then*

$$\left|\delta_2 - \lambda\delta_1^2\right| \leq \begin{cases} \frac{8(1-\alpha)}{3\pi^2}\left(\frac{1}{3} - \frac{2(1-\alpha)(3\lambda-1)}{\pi^2}\right), & \lambda \leq \frac{1}{3} - \frac{\pi^2}{36(1-\alpha)}, \\ \frac{4(1-\alpha)}{3\pi^2}, & \frac{1}{3} - \frac{\pi^2}{36(1-\alpha)} \leq \lambda \leq \frac{1}{3} + \frac{5\pi^2}{36(1-\alpha)}, \\ -\frac{8(1-\alpha)}{3\pi^2}\left(\frac{1}{3} - \frac{2(1-\alpha)(3\lambda-1)}{\pi^2}\right), & \lambda \geq \frac{1}{3} + \frac{5\pi^2}{36(1-\alpha)}. \end{cases}$$

If $\frac{1}{3} - \frac{\pi^2}{36(1-\alpha)} \leq \lambda \leq \frac{1}{3} + \frac{\pi^2}{18(1-\alpha)}$, then

$$\left|\delta_2 - \lambda\delta_1^2\right| + \left(\lambda + \frac{\pi^2}{36(1-\alpha)} - \frac{1}{3}\right)|\delta_1|^2 \leq \frac{4(1-\alpha)}{3\pi^2}.$$

Furthermore, if $\frac{1}{3}+\frac{\pi^2}{18(1-\alpha)} \le \lambda \le \frac{1}{3}+\frac{5\pi^2}{36(1-\alpha)}$, *then*

$$\left|\delta_2-\lambda\delta_1^2\right| + \left(\frac{1}{3}+\frac{5\pi^2}{36\,(1-\alpha)}-\lambda\right)|\delta_1|^2 \le \frac{4\,(1-\alpha)}{3\pi^2}.$$

Each of these results is sharp.

Proof. From (4.6), for $\lambda \in \mathbb{R}$, we get

$$\delta_2-\lambda\delta_1^2 = \frac{1}{3}\left(\gamma_2+\frac{1-3\lambda}{4}\gamma_1^2\right).$$

This implies that

$$\left|\delta_2-\lambda\delta_1^2\right| = \frac{1}{3}\left|\gamma_2-\mu\gamma_1^2\right|, \qquad \mu := \frac{3\lambda-1}{4}. \tag{4.7}$$

Using Theorem 3.3, it is obtained that

$$\left|\delta_2-\lambda\delta_1^2\right| \le \begin{cases} \frac{8(1-\alpha)}{3\pi^2}\left(\frac{1}{3}-\frac{2(1-\alpha)(3\lambda-1)}{\pi^2}\right), & \lambda \le \frac{1}{3}-\frac{\pi^2}{36(1-\alpha)}, \\ \frac{4(1-\alpha)}{3\pi^2}, & \frac{1}{3}-\frac{\pi^2}{36(1-\alpha)} \le \lambda \le \frac{1}{3}+\frac{5\pi^2}{36(1-\alpha)}, \\ -\frac{8(1-\alpha)}{3\pi^2}\left(\frac{1}{3}-\frac{2(1-\alpha)(3\lambda-1)}{\pi^2}\right), & \lambda \ge \frac{1}{3}+\frac{5\pi^2}{36(1-\alpha)}. \end{cases}$$

□

Corollary 4.4. *Let* $f \in \mathcal{UCV}$ *be given by* (1.1) *and let the coefficients of* $\log(f(z)/z)$ *be given by* (1.7). *Then*

$$\left|\delta_2-\lambda\delta_1^2\right| \le \begin{cases} \frac{4}{3\pi^2}\left(\frac{1}{3}-\frac{3\lambda-1}{\pi^2}\right), & \lambda \le \frac{1}{3}-\frac{\pi^2}{18}, \\ \frac{2}{3\pi^2}, & \frac{1}{3}-\frac{\pi^2}{18} \le \lambda \le \frac{1}{3}+\frac{5\pi^2}{18}, \\ -\frac{4}{3\pi^2}\left(\frac{1}{3}-\frac{3\lambda-1}{\pi^2}\right), & \lambda \ge \frac{1}{3}+\frac{5\pi^2}{18}. \end{cases}$$

If $\frac{1}{3}-\frac{\pi^2}{18} \le \lambda \le \frac{1}{3}+\frac{\pi^2}{9}$, *then*

$$\left|\delta_2-\lambda\delta_1^2\right| + \left(\lambda+\frac{\pi^2}{18}-\frac{1}{3}\right)|\delta_1|^2 \le \frac{2}{3\pi^2}.$$

Furthermore, if $\frac{1}{3}+\frac{\pi^2}{9} \le \lambda \le \frac{1}{3}+\frac{5\pi^2}{18}$, *then*

$$\left|\delta_2-\lambda\delta_1^2\right| + \left(\frac{1}{3}+\frac{5\pi^2}{18}-\lambda\right)|\delta_1|^2 \le \frac{2}{3\pi^2}.$$

Each of these results is sharp.

Theorem 4.5. *Let* $f \in \mathcal{UCV}(\alpha)$ $(0 \le \alpha < 1)$ *be given by* (1.1) *and let the coefficients of* $\log(f(z)/z)$ *be given by* (1.7). *Then for* $\lambda \in \mathbb{C}$,

$$\left|\delta_2-\lambda\delta_1^2\right| \le \begin{cases} \frac{8(1-\alpha)}{9\pi^4}\left|6\,(1-\alpha)\,(3\lambda-1)-\pi^2\right|, & \left|\frac{4(1-\alpha)(3\lambda-1)}{\pi^2}-\frac{2}{3}\right| \ge 1, \\ \frac{4(1-\alpha)}{3\pi^2}, & \left|\frac{4(1-\alpha)(3\lambda-1)}{\pi^2}-\frac{2}{3}\right| \le 1. \end{cases}$$

Corollary 4.6. *Let $f \in \mathcal{UCV}$ be given by* (1.1) *and let the coefficients of* $\log(f(z)/z)$ *be given by* (1.7). *Then for $\lambda \in \mathbb{C}$,*

$$\left|\delta_2 - \lambda\delta_1^2\right| \leq \begin{cases} \frac{4}{9\pi^4}\left|3(3\lambda - 1) - \pi^2\right|, & \left|\frac{2(3\lambda-1)}{\pi^2} - \frac{2}{3}\right| \geq 1, \\ \\ \frac{2}{3\pi^2}, & \left|\frac{2(3\lambda-1)}{\pi^2} - \frac{2}{3}\right| \leq 1. \end{cases}$$

4.2. Second Hankel determinant.

Theorem 4.7. *Let $f \in \mathcal{UCV}(\alpha)$ $(0 \leq \alpha < 1)$ be given by* (1.1) *and let the coefficients of* $\log(f(z)/z)$ *be given by* (1.7). *Then*

$$\left|\delta_1\delta_3 - \delta_2^2\right| \leq \frac{20(1-\alpha)^2}{9\pi^4} + \frac{64(1-\alpha)^3}{9\pi^6}.$$

Proof. From (4.6), Theorems 3.1, 3.7, and 3.3 for $\mu = 1/4$, we find

$$\begin{aligned} \left|\delta_1\delta_3 - \delta_2^2\right| &= \left|\frac{1}{8}\left(\gamma_1\gamma_3 - \gamma_2^2\right) + \frac{1}{72}\gamma_2^2 + \frac{1}{36}\gamma_1^2\left(\gamma_2 - \frac{1}{4}\gamma_1^2\right)\right| \\ &\leq \frac{1}{8}\left|\gamma_1\gamma_3 - \gamma_2^2\right| + \frac{1}{72}\left|\gamma_2\right|^2 + \frac{1}{36}\left|\gamma_1\right|^2\left|\gamma_2 - \frac{1}{4}\gamma_1^2\right| \\ &\leq \frac{2(1-\alpha)^2}{\pi^4} + \frac{2(1-\alpha)^2}{9\pi^4} + \frac{64(1-\alpha)^3}{9\pi^6} \\ &= \frac{20(1-\alpha)^2}{9\pi^4} + \frac{64(1-\alpha)^3}{9\pi^6}. \end{aligned}$$

This completes the proof. □

Corollary 4.8. *Let $f \in \mathcal{UCV}$ be given by* (1.1) *and let the coefficients of* $\log(f(z)/z)$ *be given by* (1.7). *Then*

$$\left|\delta_1\delta_3 - \delta_2^2\right| \leq \frac{5}{9\pi^4} + \frac{8}{9\pi^6}.$$

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