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LEFT AND RIGHT DRAZIN INVERTIBILITY OF OPERATOR MATRICES

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Abstract. When $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ are given, we denote by M_C the operator on the Hilbert space $\mathcal{H} \oplus \mathcal{K}$ of the form $M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$. In this paper, the closedness of ranges and left (resp. right) Drazin invertibility of upper triangular operator matrices M_C are investigated.

1. Introduction and preliminaries

Throughout this paper, let $\mathcal{H}$ and $\mathcal{K}$ be infinite-dimensional separable complex Hilbert spaces and let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ denote the set of bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. If $\mathcal{H} = \mathcal{K}$, then $\mathcal{B}(\mathcal{H}, \mathcal{H})$ will be written by $\mathcal{B}(\mathcal{H})$. For given $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$, and $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, we denote by M_C an operator acting on $\mathcal{H} \oplus \mathcal{K}$ of the form

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}. \tag{1.1}$$

Various types of invertibility and regularity have been considered in literature of an upper triangular operator matrix (1.1) as well as various types of spectra of M_C, where $\mathcal{H}, \mathcal{K}$ are separable Hilbert or Banach spaces. One can see [1–8, 14–18] and the references therein for recent reviews on this topic. This paper is devoted to the study the left and right Drazin invertibility of M_C for some $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

For $A \in \mathcal{B}(\mathcal{H})$, write $\mathcal{N}(A)$ for the kernel of A and $\mathcal{R}(A)$ for the range of A, and the ascent $a(A)$ and the descent $d(A)$ of A are given by $a(A) = \inf\{n \geq 0 : \mathcal{N}(A^n) = \mathcal{N}(A^{n+1})\}$ and $d(A) = \inf\{n \geq 0 : \mathcal{R}(A^n) = \mathcal{R}(A^{n+1})\}$. Clearly, $a(A) = 0$ if and only if A is injective, and $d(A) = 0$ if and only if A is surjective.

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If both $a(A)$ and $d(A)$ are finite, then A is said to be Drazin invertible, and its Drazin inverse is denoted by A^D. It follows from [13, Theorem 2.3] that A^D is unique and A has a unique decomposition $A_1 \oplus A_2$, where A_1 is an invertible operator and A_2 is a nilpotent one. An operator $A \in \mathcal{B}(\mathcal{H})$ is left Drazin invertible if $a(A) < \infty$ and $\mathcal{R}(A^{a(A)+1})$ is closed. An operator $A \in \mathcal{B}(\mathcal{H})$ is right Drazin invertible if $d(A) < \infty$ and $\mathcal{R}(A^{d(A)})$ is closed. The nullity and the deficiency of A are defined respectively by $\alpha(A) = \dim \mathcal{N}(A)$ and $\beta(A) = \dim \mathcal{K}/\mathcal{R}(A)$. Here I denotes the identity operator in $\mathcal{H}$.

A vast research can be found in literature devoted to the study of bounded linear operators with closed range. Their significance stems from the many applications they have, for example, in the spectral study of differential operators or in the context of perturbation theory, but also from the important role they play when it comes to purely theoretical considerations. In this paper, we prove the closedness of $\mathcal{R}(M_C^n)$ for $n \in \mathbb{N}$ which is used to give a sufficient conditions for M_C to be left (resp. right) Drazin invertible.

2. Closedness of range of the operator M_C^n

In the following section, we find the relationship of the closedness among the ranges $\mathcal{R}(A^n)$, $\mathcal{R}(B^n)$ and $\mathcal{R}(M_C^n)$ for $n \in \mathbb{N}$ in the operator matrix M_C^n. We begin with some lemmas.

Lemma 2.1 (see [10]). *If $A \in \mathcal{B}(\mathcal{H})$, then the following statements hold:*

1. *If $D \in \mathcal{B}(\mathcal{H})$, is finite rank, then $\mathcal{R}(A + D)$ is closed if and only if $\mathcal{R}(A)$ is closed.*
2. *If M and N are invertible operators, then $\mathcal{R}(MAN)$ is closed if and only if $\mathcal{R}(A)$ is closed.*
3. *If N is an invertible operator, then $\mathcal{R}(AN) = \mathcal{R}(A)$.*

Lemma 2.2 (see [10]). *If $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then the following statements are equivalent:*

1. *$\mathcal{R}(A)$ is closed.*
2. *$\mathcal{R}(AA^*)$ is closed.*
3. *$\mathcal{R}(A) = \mathcal{R}(AA^*)$.*
4. *$\mathcal{R}(A^*)$ is closed.*
5. *$\mathcal{R}(A^*A)$ is closed.*
6. *$\mathcal{R}(A^*) = \mathcal{R}(A^*A)$.*

Proposition 2.3. *Let $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{K})$ and $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be given operators such that $\mathcal{R}(A^n)$ is closed for any $n \in \mathbb{N}$ and $\beta(A^n) = \infty$. Then $\mathcal{R}(M_C^n)$ is closed.*

Proof. For any $n \in \mathbb{N}$, we have

$$M_C^n = \begin{bmatrix} A^n & S \\ 0 & B^n \end{bmatrix} : \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix} \longrightarrow \begin{bmatrix} \mathcal{H} \\ \mathcal{K} \end{bmatrix}.$$

Since $\beta(A^n)=\infty$, there exists an isometrically isomorphic linear operator $J:\mathcal{K}\to\mathcal{R}(A^n)^\perp$. Define an operator $S:\mathcal{K}\to\mathcal{H}$ by

$$S:=\begin{pmatrix} J \\ 0 \end{pmatrix}:\mathcal{K}\longrightarrow\begin{pmatrix} \mathcal{R}(A^n)^\perp \\ \mathcal{R}(A^n) \end{pmatrix}.$$

Let $\begin{pmatrix} x_k \\ y_k \end{pmatrix}_{k\in\mathbb{N}}\subset\mathcal{H}\oplus\mathcal{K}$ such that $M_C^n\begin{pmatrix} x_k \\ y_k \end{pmatrix}\longrightarrow\begin{pmatrix} u \\ v \end{pmatrix}$. Then $A^nx_k+Sy_k\to u$ and $B^ny_k\to v$. We have $Sy_k\in\mathcal{R}(A^n)^\perp$. Then $(A^nx_k+Sy_k)_{k\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{R}(A^n)+\mathcal{R}(A^n)^\perp$. Thus $(A^nx_k)_{k\in\mathbb{N}}$ and $(Sy_k)_{k\in\mathbb{N}}=(Jy_k)_{k\in\mathbb{N}}$ are two Cauchy sequences. From this we deduce that $(y_k)_{k\in\mathbb{N}}$ is a Cauchy sequence.
Let $y_k\to y_0$ and let $A^nx_k\to A^nx_0$. Then $u=A^nx_0+Sy_0$ and $v=B^ny_0$.
Hence $\begin{pmatrix} u \\ v \end{pmatrix}=M_C^n\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}\in\mathcal{R}(M_C^n)$, which means that $\mathcal{R}(M_C^n)$ is closed. □

Proposition 2.4. *Let $A\in\mathcal{B}(\mathcal{H})$, $B\in\mathcal{B}(\mathcal{K})$ and $C\in\mathcal{B}(\mathcal{K},\mathcal{H})$ be given operators such that $\mathcal{R}(B^n)$ is closed for any $n\in\mathbb{N}$ and $\alpha(B^n)=\infty$. Then $\mathcal{R}(M_C^n)$ is closed.*

Proof. We consider the operator matrix

$$M=\begin{pmatrix} B^* & C^* \\ 0 & A^* \end{pmatrix}. \tag{2.1}$$

Since $\alpha(B^n)=\beta((B^*)^n)=\infty$ and $\mathcal{R}(B^n)$ is closed, then by Proposition 2.3 $\mathcal{R}(M^n)$ is closed. On the other hand, we have

$$M_C^*=\begin{pmatrix} A^* & 0 \\ C^* & B^* \end{pmatrix}=TMT,$$

with $T=\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ and $T^{-1}=T$.
Hence

$$(M_C^*)^n=(TMT)(TMT)\dots(TMT)=TM^nT.$$

We conclude that $\mathcal{R}((M_C^*)^n)$ is closed. Thus $\mathcal{R}(M_C^n)$ is closed. □

3. Left and right Drazin invertibility of M_C

Theorem 3.1. *Let $A\in\mathcal{B}(\mathcal{H})$ be left Drazin invertible and let $B\in\mathcal{B}(\mathcal{K})$ such that*

(i) *$\beta(A^{p+1})=\infty$, with $a(A)=p$,*
(ii) *there exists $C\in\mathcal{B}(\mathcal{K},\mathcal{H})$ such that $\mathcal{N}(C)\subseteq\mathcal{N}(B)^\perp$ and $\mathcal{R}(C)\subseteq(\mathcal{R}(A)+\mathcal{N}(A^p))^\perp$.*

Then M_C is left Drazin invertible.

Proof. Since A is left Drazin invertible, then by Proposition 2.3 it follows that $\mathcal{R}(M_C^{p+1})$ is closed. Now, we prove that $a(M_C)<\infty$. It is enough to prove that $\mathcal{N}(M_C^{p+1})\subseteq\mathcal{N}(M_C^p)$. Let $\binom{x}{y}\in\mathcal{N}(M_C^{p+1})$. Then

$$\begin{cases} A^{p+1}x+A^pCy+A^{p-1}CBy+\dots+ACB^{p-1}y+CB^py=0, \\ B^{p+1}y=0, \end{cases} \tag{3.1}$$

Thus

$$A^{p+1}x + A^pCy + A^{p-1}CBy + \cdots + ACB^{p-1}y = -CB^py \in \mathcal{R}(A) \cap \left[(\mathcal{R}(A) + \mathcal{N}(A^p))^{\perp}\right] \subseteq \mathcal{R}(A) \cap \mathcal{R}(A)^{\perp} = \{0\}.$$

Hence

$$\begin{cases} A^{p+1}x + A^pCy + A^{p-1}CBy + \cdots + ACB^{p-1}y = 0, \\ B^{p+1}y = 0 \text{ and } CB^py = 0, \text{ then } B^py \in \mathcal{N}(B) \cap \mathcal{N}(C) \subseteq \mathcal{N}(B) \cap \mathcal{N}(B)^{\perp} = \{0\}. \end{cases} \tag{3.2}$$

Thus $B^py = 0$. From the first equality in (3.2) we get
$A^px + A^{p-1}Cy + A^{p-2}CBy + \cdots + CB^{p-1}y \in \mathcal{N}(A)$.
Let $A^px + A^{p-1}Cy + A^{p-2}CBy + \cdots + CB^{p-1}y = x_1$, with $x_1 \in \mathcal{N}(A)$. Then

$$\begin{cases} A^{p+1}x + A^pCy + A^{p-1}CBy + \cdots + ACB^{p-2}y - x_1 + CB^{p-1}y = 0, \\ B^py = 0. \end{cases} \tag{3.3}$$

It induces that $A^px + A^{p-1}Cy + A^{p-2}CBy + \cdots + ACB^{p-2}y - x_1 = -CB^{p-1}y \in [\mathcal{R}(A) + \mathcal{N}(A)] \cap [\mathcal{R}(A) + \mathcal{N}(A^p)]^{\perp} \subseteq [\mathcal{R}(A) + \mathcal{N}(A^p)] \cap [\mathcal{R}(A) + \mathcal{N}(A^p)]^{\perp} = \{0\}$.
This implies that $A^px + A^{p-1}Cy + A^{p-2}CBy + \cdots + ACB^{p-2}y - x_1 = -CB^{p-1}y = 0$.
Therefore $A^px + A^{p-1}Cy + A^{p-2}CBy + \cdots + ACB^{p-2}y = x_1$ and $B^{p-1}y = 0$.
Since $A^px + A^{p-1}Cy + A^{p-2}CBy + \cdots + ACB^{p-2}y = x_1$, then
$A^{p-1}x + A^{p-2}Cy + A^{p-3}CBy + \cdots + CB^{p-2}y \in \mathcal{N}(A^2)$.
Let $A^{p-1}x + A^{p-2}Cy + A^{p-3}CBy + \cdots + CB^{p-2}y = x_2$, with $x_2 \in \mathcal{N}(A^2)$. Then

$$\begin{cases} A^{p-1}x + A^{p-2}Cy + A^{p-3}CBy + \cdots + ACB^{p-3}y - x_2 + CB^{p-2}y = 0, \\ B^{p-1}y = 0. \end{cases} \tag{3.4}$$

If we continue this process, then we gets

$$\begin{cases} A^2x + ACy - x_{p-1} + CBy = 0, \\ B^2y = 0, \end{cases} \tag{3.5}$$

where $x_{p-1} \in \mathcal{N}(A^{p-1})$. Then there exists $x_p \in \mathcal{N}(A^p)$ such that

$$\begin{cases} Ax + Cy - x_p = 0, \\ By = 0. \end{cases} \tag{3.6}$$

Thus $Ax - x_p = -Cy \in [\mathcal{R}(A) + \mathcal{N}(A^p)] \cap [\mathcal{R}(A) + \mathcal{N}(A^p)]^{\perp} = \{0\}$. It follows that $x \in \mathcal{N}(A^{p+1}) = \mathcal{N}(A^p)$ and $y = 0$, so $\binom{x}{y} \in \mathcal{N}(M_C^p)$. Since $\mathcal{N}(M_C^p) \subseteq \mathcal{N}(M_C^{p+1})$ we get $a(M_C) \leq p$. □

We know that the properties to be right (resp. left) Drazin invertible are dual each other. Then we have the following result.

Theorem 3.2. *Let $A \in \mathcal{B}(\mathcal{H})$ and let $B \in \mathcal{B}(\mathcal{K})$ be right Drazin invertible such that*

(i) *$\alpha(B^k) = \infty$ with $d(B) = k$,*
(ii) *there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\mathcal{N}(A^*) \subseteq \overline{\mathcal{R}(C)}$ and $(\mathcal{N}(B) \cap \mathcal{R}(B^k))^{\perp} \subseteq \mathcal{N}(C)$.*

Then M_C is right Drazin invertible.

Proof. Since B is right Drazin invertible, then B^* is left Drazin invertible; so by Theorem 3.1, there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that

$$M = \begin{pmatrix} B^* & C^* \\ 0 & A^* \end{pmatrix} \quad \text{is left Drazin invertible.}$$

By Proposition 2.4 we have $\mathcal{R}(M_C^k)$ is closed since $\mathcal{R}(M^k)$ is closed.
Let now $k = a(M)$ and let $x \in \mathcal{N}((M_C^*)^k)$. Then $TM^kTx = 0$. It follows that $M^kTx = 0$. Hence $T(\mathcal{N}((M_C^*)^k)) \subseteq \mathcal{N}(M^k)$. On the other hand, if $x \in \mathcal{N}(M^k)$, then we obtain $Tx \in \mathcal{N}((M_C^*)^k)$. Therefore $T(Tx) \in T(\mathcal{N}((M_C^*)^k))$. That is $x \in T(\mathcal{N}((M_C^*)^k))$, which implies that $T(\mathcal{N}((M_C^*)^k)) = \mathcal{N}(M^k)$. Since $\mathcal{N}(M^k) = \mathcal{N}(M^{k+1})$, then $T(\mathcal{N}((M_C^*)^k)) = T(\mathcal{N}((M_C^*)^{k+1}))$. Thus $\mathcal{N}((M_C^*)^k) = \mathcal{N}((M_C^*)^{k+1})$. It shows that $a(M_C^*) \leq k < \infty$. Hence $d(M_C) < \infty$, and M_C is a right Drazin invertible operator. □

In the next result, we present the right Drazin invertibility of M_C via the injectivity of A.

Theorem 3.3. *Let $A \in \mathcal{B}(\mathcal{H})$ and let $B \in \mathcal{B}(\mathcal{K})$ be right Drazin invertible such that*

(i) *A is injective and $\mathcal{R}(A)$ is closed*
(ii) *there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\mathcal{N}(A^*) \subseteq \overline{\mathcal{R}(C)}$ and $(\mathcal{N}(B) \cap \mathcal{R}(B^k))^\perp \subseteq \mathcal{N}(C)$ with $d(B) = k$.*

Then M_C is right Drazin invertible.

Proof. Since B is right Drazin invertible, where $d(B) = k$ and $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ satisfies (ii), it follows that $d(M_C) = k$. Now, consider the operator M given in (2.1). If A is injective with closed range, then A^* is surjective and so is $(A^*)^k$. So, $\mathcal{R}(M^k) = \mathcal{R}((B^*)^k) \oplus \mathcal{H}$. Thus $\mathcal{R}(M^k)$ is closed, and hence $\mathcal{R}(M_C^k)$ is closed. We conclude that M_C is a right Drazin invertible operator. □

by taking adjoint in Theorem 3.3, we have the following result.

Theorem 3.4. *Let $A \in \mathcal{B}(\mathcal{H})$ be left Drazin invertible and let $B \in \mathcal{B}(\mathcal{K})$ such that*

(i) *B is surjective,*
(ii) *there exists $C \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ such that $\mathcal{N}(C) \subseteq \mathcal{N}(B)^\perp$ and $\mathcal{R}(C) \subseteq (\mathcal{R}(A) + \mathcal{N}(A^k))^\perp$.*

Then M_C is left Drazin invertible.

We end the section with an example illustrating Theorems 3.1–3.3.

Example 3.5. Let $A \in \mathcal{B}(\mathcal{H})$ be a left Drazin invertibe operator with $a(A) = p$ defined by $A = A_1 \oplus 0$, where A_1 is a left invertible operator and let $B = A^*$. Then $B = B_1 \oplus 0$ with B_1 is a right invettible operator. Let $\{f_i\}_{i=1}^\infty$ be an orthonormal basis of $\mathcal{R}(A^n)^\perp$ for $n \geq p$. Since $\mathcal{R}(A^n)$ is closed then $\beta(A^n) = \dim \mathcal{R}(A^n)^\perp$. From $\dim \mathcal{R}(A^n)^\perp = \infty$ we have $\dim(\mathcal{R}(A^n) + \mathcal{N}(A^p))^\perp = \infty$. On the other hand, $\beta(A^n) = \dim \mathcal{N}((A^*)^n) = \dim \mathcal{N}(B^n) = \infty$. There exist an isometry T

from $\mathcal{N}(B^n)$ into $(\mathcal{R}(A^n)+\mathcal{N}(A^p))^\perp$. Define $C:\mathcal{K}\to\mathcal{H}$ by

$$C=\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}:\begin{pmatrix} \mathcal{N}(B^n) \\ \mathcal{N}(B^n)^\perp \end{pmatrix}\longrightarrow\begin{pmatrix} (\mathcal{R}(A^n)+\mathcal{N}(A^p))^\perp \\ \mathcal{R}(A^n)+\mathcal{N}(A^p) \end{pmatrix}.$$

In this case, the upper-triangular operator matrix M_C has the operator matrix

$$M_C=\begin{pmatrix} A_1 & 0 & T & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & B_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Observe that $\mathcal{N}(C)=\{0\}\oplus\mathcal{N}(B^n)^\perp\sim\mathcal{N}(B^n)^\perp\subseteq\mathcal{N}(B)^\perp$ and $\mathcal{R}(C)\subseteq(\mathcal{R}(A^n)+\mathcal{N}(A^p))^\perp$. Hence $a(M_C)\leq p$, and M_C is a left Drazin invertible operator.

4. Application to a spectral boundary value matrix problem

This section is devoted to the study of boundary value problems described by an upper triangular operator matrices (2×2) acting in Hilbert spaces with a complex spectral parameter λ,

$$(\mathcal{P})\begin{cases}(U_L-\lambda M_C)w=F,\\ \Gamma w=\Phi,\end{cases}$$

where F and Φ are given and U_L is the matrix operator defined on $\mathcal{H}\oplus\mathcal{K}$ by

$$U_L=\begin{pmatrix} U_1 & L \\ 0 & U_2 \end{pmatrix},$$

with a given linear operator $L:K\longrightarrow H$. We first define the boundary value problem $(\mathcal{P})$ by ordered pairs (U_L,M_C) of an upper triangular operator matrix M_C, where U_L is right Drazin invertible, and we construct the adapted boundary operator Γ of U_L. We prove the existence of a unique solution of $(\mathcal{P})$, and we give an explicit expression for this solution. Before this down, we define the boundary operator for right Drazin invertible operator.

If A^{rd} is the right Drazin inverse of the operator A, then

$$\mathcal{R}(A^m)=\mathcal{R}(A^{rd})\oplus\mathcal{N}(A^{m+1}),\qquad \text{with } \ d(A)=m<\infty. \tag{4.1}$$

Definition 4.1 (see [12])**.** The operator $\Gamma:\mathcal{H}\to E$ is said to be an initial boundary operator for a right Drazin invertible operator A corresponding to its right Drazin inverse A^{rd} if

(i) $\Gamma A^{rd}=0$ on $\mathcal{H}$,
(ii) there exists an operator $\Pi:E\to\mathcal{H}$ such that $\Gamma\Pi=I_E$ and $\mathcal{R}(\Pi)=\mathcal{N}(A^{m+1})$ with $m=d(A)<\infty$.

Proposition 4.2 (see [11])**.** *Let $A,B\in\mathcal{B}(\mathcal{H})$. Then $(I-\lambda AB)$ is invertible if and only if $(I-\lambda BA)$ is invertible for all $\lambda\neq 0$.*

In this case, we have

$$(I-\lambda BA)^{-1}=I+\lambda B(I-\lambda AB)^{-1}A \tag{4.2}$$

and

$$(I-\lambda AB)^{-1}=I+\lambda A(I-\lambda BA)^{-1}B. \tag{4.3}$$

Corollary 4.3. *Let $A,\ B \in \mathcal{B}(\mathcal{H})$. If $\lambda^{-1} \in \rho(AB)$, then*
$$(I-\lambda AB)^{-1}A = A(I-\lambda BA)^{-1}.$$

In the following proposition, we construct the boundary operator for a Drazin invertible upper triangular matrix operator.

Proposition 4.4. *Let $U_L = \begin{pmatrix} U_1 & L \\ 0 & U_2 \end{pmatrix}$ be defined on $\mathcal{H} \oplus \mathcal{K}$. Assume that U_1^{rd} and U_2^{rd} are right Drazin inverses of U_1 and U_2, respectively. Also, Γ_1 and Γ_2 are boundary operators for U_1 and U_2 with the boundary spaces E and Z, respectively. If $\mathcal{N}(U_2^{m+1}) \subset \mathcal{N}(L^{m+1})$ with $m = \max(d(U_1), d(U_2))$, then the operator $\Gamma = \begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}$ from $\mathcal{H} \oplus \mathcal{K}$ into $E \oplus Z$ is a boundary operator for U_L.*

Proof. We observe that $\Gamma_1 U_1^{rd} = 0$, $\Gamma_2 U_2^{rd} = 0$, and there exist $\Pi_1 : E \longrightarrow \mathcal{H}$ and $\Pi_2 : Z \longrightarrow \mathcal{K}$ such that $\Gamma_1\Pi_1 = I_E$, $\mathcal{R}(\Pi_1) = \mathcal{N}(U_1^{m+1})$ and $\Gamma_2\Pi_2 = I_Z$, $\mathcal{R}(\Pi_2) = \mathcal{N}(U_2^{m+1})$. Denote by $\Pi = \begin{pmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{pmatrix} : E \oplus Z \longrightarrow X \oplus Y.$
Since U_1 and U_2 are right Drzain invertible, then so is U_L. Let U_L^{rd} the right Drazin inverse of U_L. Then $\mathcal{R}(U_L^{rd}) = \mathcal{R}(U_1^{rd}) \oplus \mathcal{R}(U_2^{rd}) \subset \mathcal{N}(\Gamma_1) \oplus \mathcal{N}(\Gamma_2) = \mathcal{N}(\Gamma)$. Hence $\Gamma U_L^{rd} = 0$ and $\Gamma\Pi = I_{E\oplus Z}$.
The condition $\mathcal{N}(U_2^{m+1}) \subset \mathcal{N}(L^{m+1})$ implies that $\mathcal{R}(\Pi) = \mathcal{N}(U_L^{m+1})$. □

Let A and B be given linear operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and consider the operator M_C defined on $\mathcal{H} \oplus \mathcal{K}$ by
$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$
where C is a linear operator from $\mathcal{H}$ into $\mathcal{K}$. According to Proposition 4.4, we define the following spectral boundary value matrix problem for unknown $w \in \mathcal{R}(U_1^m) \times \mathcal{R}(U_2^m)$ by
$$(\mathcal{P}) \begin{cases} (U_L - \lambda M_C)w = F, \\ \Gamma w = \Phi, \end{cases}$$
where $F \in \mathcal{R}(U_1^m) \times \mathcal{R}(U_2^m)$, $\Phi \in E \times Z$ and $\lambda \in \mathbb{C}$ is a spectral parameter. We denote $\mathbf{R}_\lambda[U_1^{rd}A] = (I_{\mathcal{H}} - \lambda U_1^{rd}A)^{-1}$ and $\mathbf{R}_\lambda[U_2^{rd}A] = (I_{\mathcal{K}} - \lambda U_2^{rd}B)^{-1}$, U_1^{rd} and U_2^{rd} are right Drazin inverses of U_1 and U_2, respectively.

Our purpose is to establish the existence and uniqueness of solutions for the boundary value problem $(\mathcal{P})$. In the theorem below, we give an explicit expression for the solution of the problem $(\mathcal{P})$.

Theorem 4.5. *If $\lambda^{-1} \in \rho(U_1^{rd}A) \cap \rho(U_2^{rd}B)$, then the boundary value problem $(\mathcal{P})$ is uniquely solvable for any $F \in \mathcal{H} \times \mathcal{K}$ and $\Phi \in E \times Z$ and the solution is given by*
$$w_\lambda^{F,\Phi} = G_{L,C}(U_L^{rd}F + \Pi\Phi),$$
where
$$U_L^{rd} = \begin{pmatrix} U_1^{rd} & 0 \\ 0 & U_2^{rd} \end{pmatrix}$$

and

$$G_{L,C}=\begin{pmatrix}\mathbf{R}_\lambda[U_1^{rd}A] & -U_1^{rd}\mathbf{R}_\lambda[U_1^{rd}A](L-\lambda C)\mathbf{R}_\lambda[U_2^{rd}B]\\ 0 & \mathbf{R}_\lambda[U_2^{rd}B]\end{pmatrix}.$$

Proof. We show that $(U_L-\lambda M_C)w_\lambda^{F,\Phi}=F$. We have

$$(U_L-\lambda M_C)w_\lambda^{F,\Phi}=(U_L-\lambda M_C)G_{L,C}U_L^DF+(U_L-\lambda M_C)G_{L,C}\Pi\Phi.$$

Then

$$\begin{aligned}
&(U_L-\lambda M_C)G_{L,C}U_L^{rd}F=\\
&=\ (U_L-\lambda M_C)\begin{pmatrix}\mathbf{R}_\lambda[U_1^{rd}A] & -U_1^{rd}\mathbf{R}_\lambda[U_1^{rd}A](L-\lambda C)\mathbf{R}_\lambda[U_2^{rd}B]\\ 0 & \mathbf{R}_\lambda[U_2^{rd}B]\end{pmatrix}\begin{pmatrix}U_1^{rd}f_1\\ U_2^{rd}f_2\end{pmatrix}\\
&=\ \begin{pmatrix}(U_1-\lambda A) & (L-\lambda C)\\ 0 & (U_2-\lambda B)\end{pmatrix}\\
&\quad\times\begin{pmatrix}\mathbf{R}_\lambda[U_1^{rd}A]U_1^{rd}f_1-U_1^{rd}\mathbf{R}_\lambda[U_1^{rd}A](L-\lambda C)\mathbf{R}_\lambda[U_2^{rd}B]U_2^{rd}f_2\\ \mathbf{R}_\lambda[U_2^{rd}B]U_2^{rd}f_2\end{pmatrix}\\
&=\ \begin{pmatrix}(U_1-\lambda A)U_1^{rd}\mathbf{R}_\lambda[AU_1^{rd}]f_1\\ (U_2-\lambda B)U_2^{rd}\mathbf{R}_\lambda[BU_2^{rd}]f_2\end{pmatrix}=F,
\end{aligned}$$

and

$$\begin{aligned}
&(U_L-\lambda M_C)G_{L,C}\Pi\Phi=\\
&=\ \begin{pmatrix}(U_1-\lambda A)[\mathbf{R}_\lambda[U_1^{rd}A]\Pi_1\varphi_1-U_1^{rd}\mathbf{R}_\lambda[U_1^{rd}A](L-\lambda C)\mathbf{R}_\lambda[U_2^{rd}B]\Pi_2\varphi_2]\\ +(L-\lambda C)\mathbf{R}_\lambda[U_2^{rd}B]\Pi_2\varphi_2\\ (U_2-\lambda B)\mathbf{R}_\lambda[U_2^{rd}B]\Pi_2\varphi_2\end{pmatrix}\\
&=\ \begin{pmatrix}(U_1-\lambda A)\mathbf{R}_\lambda[U_1^{rd}A]\Pi_1\varphi_1\\ (U_2-\lambda B)\mathbf{R}_\lambda[U_2^{rd}B]\Pi_2\varphi_2\end{pmatrix}\\
&=\ \begin{pmatrix}(U_1-\lambda A)[I_{\mathcal{H}}+\lambda U_1^{rd}\mathbf{R}_\lambda[AU_1^{rd}]A]\Pi_1\varphi_1\\ (U_2-\lambda B)[I_{\mathcal{K}}+\lambda U_2^{rd}\mathbf{R}_\lambda[BU_2^{rd}]B]\Pi_2\varphi_2\end{pmatrix}\\
&=\ \begin{pmatrix}(U_1-\lambda A)\Pi_1\varphi_1+\lambda A\Pi_1\varphi_1\\ (U_2-\lambda B)\Pi_2\varphi_2+\lambda B\Pi_2\varphi_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},
\end{aligned}$$

since $\mathcal{R}(\Pi_1)=\mathcal{N}(U_1^{m+1})$ and $\mathcal{R}(\Pi_2)=\mathcal{N}(U_2^{m+1})$.
Using the fact that $\Gamma_1U_1^{rd}=0$ and $\Gamma_2U_2^{rd}=0$, we get

$$\begin{aligned}
\Gamma w_\lambda^{F,\Phi}\ &=\ \Gamma G_{L,C}(U_L^{rd}F+\Pi\Phi)\\
&=\ \begin{pmatrix}\Gamma_1 & 0\\ 0 & \Gamma_2\end{pmatrix}\begin{pmatrix}\mathbf{R}_\lambda[U_1^{rd}A]U_1^{rd}f_1-U_1^{rd}\mathbf{R}_\lambda[U_1^{rd}A](L-\lambda C)\mathbf{R}_\lambda[U_2^{rd}B]U_2^{rd}f_2\\ \mathbf{R}_\lambda[U_2^{rd}B]U_2^{rd}f_2\end{pmatrix}\\
&\quad+\begin{pmatrix}\Gamma_1 & 0\\ 0 & \Gamma_2\end{pmatrix}\begin{pmatrix}\mathbf{R}_\lambda[U_1^{rd}A]\Pi_1\varphi_1-U_1^{rd}\mathbf{R}_\lambda[U_1^{rd}A](L-\lambda C)\mathbf{R}_\lambda[U_2^{rd}B]\Pi_2\varphi_2\\ \mathbf{R}_\lambda[U_2^{rd}B]\Pi_2\varphi_2\end{pmatrix}\\
&=\ \begin{pmatrix}\Gamma_1\mathbf{R}_\lambda[U_1^{rd}A]\Pi_1\varphi_1-\Gamma_1U_1^{rd}\mathbf{R}_\lambda[U_1^{rd}A](L-\lambda C)\mathbf{R}_\lambda[U_2^{rd}B]\Pi_2\varphi_2\\ \Gamma_2\mathbf{R}_\lambda[U_2^{rd}B]\Pi_2\varphi_2\end{pmatrix}\\
&=\ \begin{pmatrix}\Gamma_1[I_{\mathcal{H}}+\lambda U_1^{rd}\mathbf{R}_\lambda[AU_1^{rd}]A]\Pi_1\varphi_1\\ \Gamma_2[I_{\mathcal{K}}+\lambda U_2^{rd}\mathbf{R}_\lambda[BU_2^{rd}]B]\Pi_2\varphi_2\end{pmatrix}\\
&=\ \begin{pmatrix}\Gamma_1\Pi_1\varphi_1\\ \Gamma_2\Pi_2\varphi_2\end{pmatrix}=\Phi.
\end{aligned}$$

The uniqueness of the solution of $(\mathcal{P})$ follows from standard arguments. That is, if $w_1, w_2 \in \mathcal{R}(U_1^m) \times \mathcal{R}(U_2^m)$ are two solutions of $(\mathcal{P})$, then $w_0 = w_1 - w_2 = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \begin{pmatrix} U_1^{rd} f_0 + \Pi_1 \varphi_0 \\ U_2^{rd} g_0 + \Pi_2 \psi_0 \end{pmatrix}$ for $(f_0, g_0) \in \mathcal{R}(U_1^m) \times \mathcal{R}(U_2^m)$, $\varphi_0 \in E$ and $\psi_0 \in Z$. Thus,

$$\begin{cases} (U_L - \lambda M_C) w_0 = 0, \\ \Gamma w_0 = 0. \end{cases}$$

Since $\Gamma_1 U_1^{rd} = 0, \Gamma_2 U_2^{rd} = 0$ and $\Gamma \Pi = I_{E \oplus Z}$, we deduce that $\begin{pmatrix} \varphi_0 \\ \psi_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then $u_0 = U_1^{rd} f_0$ and $v_0 = U_2^{rd} g_0$. So,

$$\begin{aligned} 0 = (U_L - \lambda M_C) w_0 &= \begin{pmatrix} (U_1 - \lambda A) & (L - \lambda C) \\ 0 & (U_2 - \lambda B) \end{pmatrix} \begin{pmatrix} U_1^{rd} f_0 \\ U_2^{rd} g_0 \end{pmatrix} \\ &= \begin{pmatrix} (U_1 - \lambda A) U_1^{rd} f_0 + (L - \lambda C) U_2^{rd} g_0 \\ (U_2 - \lambda B) U_2^{rd} g_0 \end{pmatrix}. \end{aligned}$$

Then, $f_0 = g_0 = 0$, since $\lambda^{-1} \in \rho(U_1^{rd} A) \cap \rho(U_2^{rd} B)$. Hence $w_1 = w_2$ and the uniqueness is proved. □

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