Khayyam J. Math. 10 (2024), no. 1, 10-30
DOI: 10.22034/KJM.2023.386475.2778

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

COMPLEX INTERPOLATION OF SOME BANACH SPACES INCLUDING MORREY SPACES

NOUFFOU DIARRA[1*] AND IBRAHIM FOFANA[1]

Communicated by H.R. Ebrahimi Vishki

Abstract. Let $1 \leq q \leq \alpha < \infty$. The sets $\left\{(L^q, l^p)^\alpha(\mathbb{R}^d) : \alpha \leq p \leq \infty\right\}$ and $\left\{F(q,p,\alpha)(\mathbb{R}^d) : \alpha \leq p \leq \infty\right\}$ are two nondecreasing families of Banach spaces such that, for both, the Lebsegue space $L^\alpha(\mathbb{R}^d)$ is the minimal element and the Morrey space $\mathcal{M}_q^\alpha(\mathbb{R}^d)$ is the maximal element. It is also known that for $1 < q \leq \alpha \leq p \leq \infty$, $(L^q, l^p)^\alpha(\mathbb{R}^d)$ has a predual space $\mathcal{H}(q', p', \alpha')(\mathbb{R}^d)$, which is also its Köthe dual space when $\alpha < p$. In this paper, we obtain complex interpolation theorems in the above mentioned three families of Banach spaces. Our results extend analogous ones recently obtained for Morrey spaces and their preduals.

1. Introduction and main results

For $1 \leq q \leq \alpha \leq \infty$, the Morrey space $\mathcal{M}_q^\alpha = \mathcal{M}_q^\alpha(\mathbb{R}^d)$, introduced in 1938 by Morrey [12] in connexion with regularity problems of solutions to partial differential equations, is defined as the set of all elements f of $L^q_{\mathrm{loc}}(\mathbb{R}^d)$ for which

$$\|f\|_{\mathcal{M}_q^\alpha} = \sup_{x\in\mathbb{R}^d,\, r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)} \left(\int_{Q(x,r)} |f(y)|^q dy\right)^{\frac{1}{q}} < \infty$$

with

$$Q(x,r) = \prod_{j=1}^{d} \left[x_j - \frac{r}{2}, x_j + \frac{r}{2}\right), \qquad x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \text{ and } 0 < r < \infty.$$

Date: Received: 19 February 2023; Revised: 4 August 2023; Accepted: 8 January 202.
*Corresponding author.
2020 *Mathematics Subject Classification.* Primary 46B70; Secondary 42B35, 46E30.
Key words and phrases. Complex interpolation, Morrey spaces, predual, normed Köthe space, Banach lattice.

Note that if $q = \alpha$, then $\mathcal{M}_q^\alpha$ coincides with the classical Lebesgue space $L^\alpha = L^\alpha(\mathbb{R}^d)$. However $\mathcal{M}_q^\alpha$ is strictly larger than L^α when $q < \alpha$.

For $1 \leq q \leq \alpha \leq p \leq \infty$, the spaces $(L^q, l^p)^\alpha$ and $F(q,p,\alpha)$ (see Section 2 for their definitions) have been introduced since 1988 and 2015, respectively, [6,8]. They arise naturally in the study of Fourier multipliers and boundedness properties of Riesz potential operators.

It is well known that, if α belongs to $\{q,p\}$, then both $(L^q, l^p)^\alpha$ and $F(q,p,\alpha)$ coincide with the Lebesgue space L^α, and when $p = \infty$, they coincide with the Morrey space $\mathcal{M}_q^\alpha$. However, if $q < \alpha < p$, then the following strict inclusions hold:

$$L^\alpha \subsetneq F(q,p,\alpha) \subsetneq (L^q, l^p)^\alpha \subsetneq \mathcal{M}_q^\alpha.$$

Let us recall that, for $1 < q \leq \alpha \leq p \leq \infty$, a predual space of $(L^q, l^p)^\alpha$ denoted by $\mathcal{H}(q', p', \alpha')$ (see Section 2 for its definition) has been described by Feichtinger and Feuto [5], where for $1 \leq s \leq \infty$, s' denotes the conjugate exponent of s, $\frac{1}{s'} = 1 - \frac{1}{s}$ with the convention $\frac{1}{\infty} = 0$. In [4], we proved that $\mathcal{H}(q', 1, \alpha')$ coincides with the so-called block space $\mathcal{B}_{q'}^{\alpha'}$ defined in [2] and which represents a predual space of the Morrey space $\mathcal{M}_q^\alpha$.

Many classical results for Lebesgue and Morrey spaces have been obtained in the framework of the spaces $(L^q, l^p)^\alpha$ and $F(q,p,\alpha)$ (see [4,8] and the references therein). Although the complex interpolation spaces of Lebesgue and Morrey spaces and their preduals are known, those of $(L^q, l^p)^\alpha$-spaces, $F(q,p,\alpha)$-spaces, and $\mathcal{H}(q', p', \alpha')$-spaces are still unknown when $1 < q < \alpha < p < \infty$.

Note that the interpolation theory is a very useful tool in the study of boundedness properties of operators in various spaces.

The main purpose of the present paper is to describe complex interpolation spaces of $(L^q, l^p)^\alpha$-spaces, $F(q,p,\alpha)$-spaces, and $\mathcal{H}(q', p', \alpha')$-spaces. In doing so, we will use some properties of Banach lattices, which have been shown to be useful in the description of complex interpolation spaces (see [10,14,17]).

Let us recall some recent results about the interpolation of Morrey spaces. For instance, as far as Calderón's first and second complex interpolation functors $(Y_0, Y_1) \mapsto [Y_0, Y_1]_\theta$ and $(Y_0, Y_1) \mapsto [Y_0, Y_1]^\theta$ (see [3] for their definitions) are concerned, the following description of interpolations spaces of Morrey spaces and their preduals are known.

Let $0 < \theta < 1$, $1 \leq q_j \leq \alpha_j < \infty$ for j in $\{0,1\}$, $q_0 \neq q_1$, and $q_0\alpha_1 = \alpha_0 q_1$. Define

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \text{ and } \frac{1}{\alpha} = \frac{1-\theta}{\alpha_0} + \frac{\theta}{\alpha_1}.$$

In 2014, Lemarié-Rieusset [11] proved that

$$\left[\mathcal{M}_{q_0}^{\alpha_0}, \mathcal{M}_{q_1}^{\alpha_1}\right]^\theta = \mathcal{M}_q^\alpha.$$

Later on, in 2020, Hakim [9] showed that

$$\left[\mathcal{M}_{q_0}^{\alpha_0}, \mathcal{M}_{q_1}^{\alpha_1}\right]_\theta = \left\{ f \in \mathcal{M}_q^\alpha : \lim_{n\to\infty} \left\| f - f\chi_{\{\frac{1}{n} \leq |f| \leq n\}} \right\|_{\mathcal{M}_q^\alpha} = 0 \right\}.$$

If in addition $1 < q_j$ for j in $\{0,1\}$, then it is true that

$$\left[\mathring{\mathcal{M}}^{\alpha_0}_{q_0}, \mathring{\mathcal{M}}^{\alpha_1}_{q_1}\right]_\theta = \left[\mathring{\mathcal{M}}^{\alpha_0}_{q_0}, \mathcal{M}^{\alpha_1}_{q_1}\right]_\theta = \left[\mathcal{M}^{\alpha_0}_{q_0}, \mathring{\mathcal{M}}^{\alpha_1}_{q_1}\right]_\theta = \mathring{\mathcal{M}}^{\alpha}_{q},$$

where $\mathring{\mathcal{M}}^v_u$ denotes the closure in $\mathcal{M}^v_u$ of the set $\mathcal{C}^\infty_c = \mathcal{C}^\infty_c(\mathbb{R}^d)$ of all infinitely differentiable and compactly supported functions on $\mathbb{R}^d$ (see [15]). Furthermore, Yuan [16] proved that

$$\left[\mathcal{B}^{\alpha_0'}_{q_0'}, \mathcal{B}^{\alpha_1'}_{q_1'}\right]_\theta = \left[\mathcal{B}^{\alpha_0'}_{q_0'}, \mathcal{B}^{\alpha_1'}_{q_1'}\right]^\theta = \mathcal{B}^{\alpha'}_{q'}.$$

In this paper, we are interested in the description of Calderón's, first, and second, complex interpolation spaces $[X_0, X_1]_\theta$ and $[X_0, X_1]^\theta$, where $0 < \theta < 1$ and the spaces X_j $(j = 0,1)$ are both in $\{(L^q, l^p)^\alpha,\ 1 \leq q \leq \alpha \leq p \leq \infty\}$, in $\{F(q,p,\alpha),\ 1 \leq q \leq \alpha \leq p \leq \infty\}$, or in $\{\mathcal{H}(q',p',\alpha'),\ 1 < q \leq \alpha \leq p \leq \infty\}$.

Our main results are the following theorems, which extend the above results.

Theorem 1.1. *Let us assume the following hypotheses:*

(i) $1 \leq q_j \leq \alpha_j \leq p_j \leq \infty$ *with* $\alpha_j < \infty$ *for* j *in* $\{0,1\}$ *and* $q_0 \neq q_1$,
(ii) $\frac{\alpha_0}{\alpha_1} = \frac{q_0}{q_1} = \frac{p_0}{p_1}$,
(iii) $0 < \theta < 1$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $\frac{1}{\alpha} = \frac{1-\theta}{\alpha_0} + \frac{\theta}{\alpha_1}$ *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$,
(iv) $\Big(X_0, X_1, X\Big)$ *is equal to* $\Big((L^{q_0}, l^{p_0})^{\alpha_0}, (L^{q_1}, l^{p_1})^{\alpha_1}, (L^q, l^p)^\alpha\Big)$ *or* $\Big(F(q_0,p_0,\alpha_0), F(q_1,p_1,\alpha_1), F(q,p,\alpha),\Big)$.

Then

$$[X_0, X_1]_\theta = \left\{ f \in X :\ \lim_{n\to\infty} \left\| f\chi_{\mathbb{R}^d \setminus \left\{\frac{1}{n} < |f| < n\right\}} \right\|_X = 0 \right\}.$$

Theorem 1.2. *Let us assume that the hypotheses (i), (iii), and (iv) of Theorem 1.1 are satisfied and that* $\frac{\alpha_0}{\alpha_1} = \frac{q_0}{q_1} \leq \frac{p_0}{p_1}$. *Then*

$$\left[\mathring{X}_0\,, \mathring{X}_1\right]_\theta = \left[\mathring{X}_0\,, X_1\right]_\theta = \left[X_0\,, \mathring{X}_1\right]_\theta = \mathring{X},$$

where $\mathring{Y}$ *denotes the closure in* Y *of* $\mathcal{C}^\infty_c$.

From Theorems 1.1 and 1.2, it is clear that if X is $(L^q, l^p)^\alpha$ or $F(q,p,\alpha)$, then $\mathring{X}$ is included in $\ddot{X} = \left\{ f \in X :\ \lim_{n\to\infty} \left\| f\chi_{\mathbb{R}^d \setminus \left\{\frac{1}{n} < |f| < n\right\}} \right\|_X = 0 \right\}$. It is worth noting that the inclusions $\mathring{X} \subset \ddot{X} \subset X$ may be strict (see Proposition 3.15).

Theorem 1.3. *Let us assume the following hypotheses:*

(i) $1 < q_j \leq \alpha_j < p_j \leq \infty$ *for* j *in* $\{0,1\}$ *and* $q_0 \neq q_1$,
(ii) $\frac{\alpha_0}{\alpha_1} = \frac{q_0}{q_1} \leq \frac{p_0}{p_1}$,
(iii) $0 < \theta < 1$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $\frac{1}{\alpha} = \frac{1-\theta}{\alpha_0} + \frac{\theta}{\alpha_1}$ *and* $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$.

Then

$$[\mathcal{H}(q_0',p_0',\alpha_0'), \mathcal{H}(q_1',p_1',\alpha_1')]_\theta = [\mathcal{H}(q_0',p_0',\alpha_0'), \mathcal{H}(q_1',p_1',\alpha_1')]^\theta = \mathcal{H}(q',p',\alpha').$$

Theorem 1.4. *Keep the same assumptions as in Theorem 1.3. Then*

$$[(L^{q_0}, l^{p_0})^{\alpha_0}\,, (L^{q_1}, l^{p_1})^{\alpha_1}]^\theta = (L^q, l^p)^\alpha.$$

The remainder of this paper is organized as follows. Section 2 is dedicated to Definitions. In Section 3, we prove some complementary results on normed Köthe spaces and recall some useful properties of Calderón product of Banach lattices and Calderón's complex interpolation functors. We also prove some auxiliary results on $(L^q, l^p)^\alpha$-spaces, $F(q,p,\alpha)$-spaces, and $\mathcal{H}(q',p',\alpha')$-spaces. Section 4 is devoted to the proofs of our main results.

2. Definitions

Let $L^0 = L^0(\mathbb{R}^d)$ denote the set of equivalence classes (modulo equality almost everywhere) of measurable functions on $\mathbb{R}^d$. Moreover, $|A|$ and χ_A stand for the Lebesgue measure and the characteristic function of the subset A of $\mathbb{R}^d$. For $1 \leq q \leq \infty$, $\|\cdot\|_q$ denotes the usual norm of the classical Lebesgue space $L^q = L^q(\mathbb{R}^d)$.

Notation 2.1. *Let r be an element of $(0,\infty)$. We set*

- $I_k^r = \prod_{j=1}^{d} [k_j r, (k_j+1)r)\,, \qquad k = (k_1, k_2, \dots, k_d) \in \mathbb{Z}^d,$
- $\mathcal{Q} = \left\{Q(x,r) : \ (r,x) \in (0,\infty) \times \mathbb{R}^d\right\},$
- $\mathcal{P} = \{\{Q_i\}_{i\in\mathbf{I}} \subset \mathcal{Q} : \ \mathbf{I}$ *is countable and* $Q_i \cap Q_j = \varnothing$ *if* $i \neq j\}$.

Definition 2.2. Let us assume that $1 \leq q, p, \alpha \leq \infty$.

(1) $$(L^q, l^p)^\alpha = (L^q, l^p)^\alpha(\mathbb{R}^d) = \{f \in L^0 : \ \|f\|_{q,p,\alpha} < \infty\},$$

where

$$||f||_{q,p,\alpha} = \sup_{r>0} r^{d\left(\frac{1}{\alpha}-\frac{1}{q}\right)}\ {}_r||f||_{q,p} \tag{2.1}$$

with

$${}_r\|f\|_{q,p} = \begin{cases} \left(\displaystyle\sum_{k\in\mathbb{Z}^d} \|f\chi_{I_k^r}\|_q^p\right)^{\frac{1}{p}} & \text{if } \ p<\infty, \\ \displaystyle\sup_{k\in\mathbb{Z}^d} \|f\chi_{I_k^r}\|_q & \text{if } \ p=\infty\ . \end{cases}$$

(2) $$F(q,p,\alpha) = F(q,p,\alpha)(\mathbb{R}^d) = \{f \in L^0 : \ \|f\|_{F(q,p,\alpha)} < \infty\},$$

where

$$\|f\|_{F(q,p,\alpha)} = \begin{cases} \displaystyle\sup_{\{Q_i\}\in\mathcal{P}} \left[\sum_{i\in\mathbf{I}} \left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}} \|f\chi_{Q_i}\|_q\right)^p\right]^{\frac{1}{p}} & \text{if } \ p<\infty, \\ \displaystyle\sup_{Q\in\mathcal{Q}} |Q|^{\frac{1}{\alpha}-\frac{1}{q}} \|f\chi_Q\|_q & \text{if } \ p=\infty\ . \end{cases} \tag{2.2}$$

For $1 \leq q \leq \alpha \leq \infty$, it is noted in [7, 8] that $\{(L^q, l^p)^\alpha : \alpha \leq p \leq \infty\}$ and $\{F(q,p,\alpha) : \alpha \leq p \leq \infty\}$ are two nondecreasing families (with respect to inclusion) of Banach spaces such that

- $F(q,p,\alpha) = (L^q, l^p)^\alpha = \{0\}$ if $\alpha \notin [q,p]$,

- $F(q,p,\alpha) = (L^q, l^p)^\alpha = L^\alpha \quad$ if $\alpha \in \{q,p\}$,
- $F(q,\infty,\alpha) = (L^q, l^\infty)^\alpha = \mathcal{M}_q^\alpha$,
- if $q < \alpha < p < \infty$, then

$$L^\alpha \subsetneq F(q,p,\alpha) \subsetneq (L^q, l^p)^\alpha \subsetneq \mathcal{M}_q^\alpha. \tag{2.3}$$

We recall below the definition of Wiener amalgam spaces.

Definition 2.3. (1) The Wiener amalgam space (L^q, l^p) $(1 \leq q,p \leq \infty)$ is defined by

$$(L^q, l^p) = (L^q, l^p)(\mathbb{R}^d) = \left\{ f \in L^0(\mathbb{R}^d) : \ {}_1\|f\|_{q,p} < \infty \right\}.$$

(2) For $1 \leq q \leq \infty$, $(L^q, \mathfrak{c}_0) = \left\{ f \in (L^q, l^\infty)(\mathbb{R}^d) \ : \ \lim_{|k| \to \infty} \|f \chi_{I_k^1}\|_q = 0 \right\}$.

It is well known that for $1 \leq q,p \leq \infty$, $((L^q, l^p), {}_1\|\cdot\|_{q,p})$ is a Banach space in which, when $1 \leq q \leq \alpha \leq p \leq \infty$, $(L^q, l^p)^\alpha)$ and therefore $F(q,p,\alpha)$ is included (see [7] and (2.3)).

Definition 2.4. Let us assume that $1 \leq q \leq \alpha \leq p \leq \infty$.

(1) For $\alpha < \infty$, the dilation operator $St_\rho^{(\alpha)}$ is an isometric operator defined by

$$St_\rho^{(\alpha)} f = \rho^{-\frac{d}{\alpha}} f\left(\rho^{-1}.\right), \qquad f \in L^0, \ \ 0 < \rho < \infty.$$

(2) A sequence $\{(c_n, \rho_n, f_n)\}_{n \geq 1}$ of elements of $\mathbb{C} \times (0,\infty) \times (L^{q'}, l^{p'})$ is called an $\mathfrak{h}$-decomposition of an element f of L^0 if

$$\begin{cases} {}_1\|f_n\|_{q',p'} \leq 1, \qquad n \geq 1, \\ \displaystyle\sum_{n \geq 1} |c_n| < \infty, \\ f = \displaystyle\sum_{n \geq 1} c_n St_{\rho_n}^{(\alpha')} f_n \quad in \ \ L^0 . \end{cases}$$

(3) The space $\mathcal{H}(q',p',\alpha') = \mathcal{H}(q',p',\alpha')(\mathbb{R}^d)$ is defined as the set of all elements of L^0 whose set of $\mathfrak{h}$-decompositions is nonvoid; in other words,

$$\mathcal{H}(q',p',\alpha') = \{ f \in L^0 : \quad \|f\|_{\mathcal{H}(q',p',\alpha')} < \infty \}$$

with

$$\|f\|_{\mathcal{H}(q',p',\alpha')} = \inf \left\{ \sum_{n \geq 1} |c_n| \right\},$$

where the infimum is taken over all $\mathfrak{h}$-decompositions of f with the convention $\inf \varnothing = \infty$.

3. Preliminaries

3.1. **Normed Köthe spaces.** We denote by L^0_+ the set of all nonnegative elements of L^0. Following Zaanen [18], we adopt the definitions below.

Definition 3.1. (1) A function norm on $\mathbb{R}^d$ is a map σ of L^0_+ into $[0,\infty]$ such that, for any elements f and g of L^0_+ and any real number $a \geq 0$, we have
   (i) $\sigma(f) = 0 \iff f = 0$ in L^0,
   (ii) $\sigma(af) = a\,\sigma(f)$,
   (iii) $\sigma(f+g) \leq \sigma(f) + \sigma(g)$,
   (iv) $f \leq g$ in $L^0 \Longrightarrow \sigma(f) \leq \sigma(g)$.

(2) If σ is a function norm on $\mathbb{R}^d$, then $L^\sigma = \{f \in L^0 : \ \sigma(|f|) < \infty\}$ is called the normed Köthe space on $\mathbb{R}^d$ defined by σ.

(3) A Banach function space on $\mathbb{R}^d$ is a Banach space $(B, \|\cdot\|_B)$ such that
   (i) B is a linear subspace of L^0,
   (ii) there is a function norm σ on $\mathbb{R}^d$ such that $B = L^\sigma$ and

$$\|f\|_B = \sigma(|f|), \qquad f \in B.$$

Definition 3.2. Let L be a normed Köthe space on $\mathbb{R}^d$ defined by a function norm σ and $\|\cdot\|_L = \sigma(|\cdot|)$.

(1) An element f of L is of absolutely continuous norm whenever $\lim_{n\to\infty} \|f_n\|_L = 0$ for every sequence $(f_n)_{n\geq 1}$ in L such that

$$\begin{cases} |f| \geq f_n \geq f_{n+1} \quad \text{in} \ \ L^0, \quad n \geq 1, \\ \\ \lim_{n\to\infty} f_n = 0 \ \ \text{in} \ \ L^0. \end{cases}$$

(2) L satisfies the Fatou property if for any sequence $(f_n)_{n\geq 1}$ of elements of L

$$. \left[0 \leq f_n \uparrow f \ \text{ in } \ L^0 \ \text{ and } \sup_{n\geq 1} \|f_n\|_L < \infty\right] \Longrightarrow \left[f \in L \ \text{ and } \lim_{n\to\infty} \|f_n\|_L = \|f\|_L\right].$$

(3) An order ideal in L is a linear subspace A of L such that

$$\left[g \in A,\ f \in L^0 \text{ and } |f| \leq |g| \ \text{ in } L^0\right] \Longrightarrow \ f \in A.$$

We adopt the following notations.

Notation 3.3. *Let L be a normed Köthe space.*

(1) $L_a = \{f \in L : \ f$ *is of absolutely continuous norm*$\}$.

(2) $\ddot{L} = \left\{f \in L : \lim_{n\to\infty} \left\| f\chi_{\mathbb{R}^d \setminus \{\frac{1}{n} < |f| < n\}} \right\|_L = 0\right\}$.

(3) $\mathring{L}$ *denotes the closure in L of the set $\mathcal{C}_c^\infty$ of all infinitely differentiable and compactly supported functions on $\mathbb{R}^d$.*

We prove the following proposition.

Proposition 3.4. *Let L be a normed Köthe space. Then*

(1) *L_a and $\ddot{L}$ are closed order ideals in L,*

(2) *L_a is included in $\ddot{L}$,*

(3) *if E is a measurable subset of $\mathbb{R}^d$ such that χ_E is in L, then χ_E belongs to $\ddot{L}$.*

Proof. (1) • For L_a, the result is contained in [18, Theorem 3, § 72].
• It is easy to show that $\ddot{L}$ is an order ideal in L.
• It is proved in [9, Lemma 4.5] that $\ddot{\mathcal{M}}_q^\alpha$ is a closed subspace of $\mathcal{M}_q^\alpha$. We may follow word for word the argumentation used there to show that $\ddot{L}$ is a closed subspace of L.

(2) Let f be an element of L_a, and set, for any positive integer n, $f_n = f\chi_{\mathbb{R}^d\setminus\{\frac{1}{n}<|f|<n\}}$.
• It is clear that f_n $(n \geq 1)$ are in L and satisfy

$$|f| \geq |f_n| \geq |f_{n+1}| \quad \text{in } L^0, \qquad n \geq 1.$$

• Set $E = \left\{x \in \mathbb{R}^d : \ |f(x)| < \infty\right\}$.
Since f is in the normed Köthe space L, $\left|\mathbb{R}^d \setminus E\right| = 0$ (see [18, Theorem 1, § 63]). Let x be an element of E.

First case: $f(x) = 0$.
We have for any positive integer n, $f_n(x) = 0$ and so $\lim\limits_{n\to\infty} f_n(x) = 0$.

Second case: $0 < |f(x)| < \infty$.
There exists a positive integer n_x such that $\frac{1}{n_x} < |f(x)| < n_x$, and therefore for any positive integer $n \geq n_x$, $f_n(x) = 0$ and so $\lim\limits_{n\to\infty} f_n(x) = 0$. Hence $(|f_n|)_{n\geq 1}$ converges to 0 in L^0.
• From what proceeds and since f is in L_a , $\lim\limits_{n\to\infty} \|f_n\|_L = 0$, and so f belongs to $\ddot{L}$.

(3) Let E be a measurable subset of $\mathbb{R}^d$ such that χ_E belongs to L. We have

$$\mathbb{R}^d \setminus \left\{\frac{1}{n} < \chi_E < n\right\} = \mathbb{R}^d \setminus E, \qquad n \geq 1,$$

and therefore

$$\chi_E\, \chi_{\mathbb{R}^d\setminus\left\{\frac{1}{n}<\chi_E<n\right\}} = \chi_E \chi_{\mathbb{R}^d\setminus E} = 0, \qquad n \geq 1.$$

Consequently,

$$\lim_{n\to\infty} \|\chi_E\, \chi_{\mathbb{R}^d\setminus\left\{\frac{1}{n}<\chi_E<n\right\}}\|_L = 0.$$

This proves that χ_E belongs to $\ddot{L}$. □

The following result is a generalization of point b) of Proposition 4.6 in our paper [4]. The proof is obtained by almost the same argumentation used there.

Proposition 3.5. *Let us assume that L is a normed Köthe space, that $1 \leq \alpha < \infty$, and that the Lebesgue space L^α is continuously included in L. Then L_a is equal to the closure in L of L^α, and therefore $L_a = \mathring{L}$.*

Proof. • Let f be any element of L_a, and set

$$f_n = \operatorname{sgn}(f) \min\left\{|f|, n\chi_{Q(0,2n)}\right\}, \qquad n \geq 1,$$

where

$$\operatorname{sgn}(f)(x) = \begin{cases} \frac{f(x)}{|f(x)|} & \text{if } f(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\begin{cases} |f_n| \text{ is in } L^\alpha \text{ and } |f_n| \leq |f| \text{ in } L^0, \quad n \geq 1, \\ \lim_{n\to\infty} f_n = f \quad \text{in } L^0. \end{cases}$$

Since f is of absolutely continuous norm in L, $\lim_{n\to\infty} \|f - f_n\|_L = 0$ (see [18, Theorem 2 § 72]), and therefore f is in the closure $\overline{L^\alpha}$ of L^α in L.

- Let f be any element of L^α and let $\{E_n\}_{n\geq 1}$ be a nonincreasing sequence of measurable subsets of $\mathbb{R}^d$ such that $\left|\bigcap_{n\geq 1} E_n\right| = 0$. It is clear that we can apply the classical dominated convergence theorem for Lebesgue spaces to obtain $\lim_{n\to\infty} \|f\chi_{E_n}\|_\alpha = 0$ and, since L^α is continuously embedded in L, $\lim_{n\to\infty} \|f\chi_{E_n}\|_L = 0$. This shows that f is in L_a (see [18, Theorem 1, § 72]).
- We have proved that $L^\alpha \subset L_a \subset \overline{L^\alpha}$.

Since L_a is closed in L (see Proposition 3.4), we obtain $L_a = \overline{L^\alpha}$. Therefore, since $\mathcal{C}_c^\infty$ is dense in L^α and L^α is continuously embedded in L, we can conclude that $L_a = \mathring{L}$. □

Let L be a normed Köthe space. The Köthe dual space (or associate space) L' of L is defined as the set of all elements g of L^0 such that

$$\|g\|_{L'} = \sup\left\{\int_{\mathbb{R}^d} |g(x)f(x)|\, dx : \quad f \in L \text{ and } \|f\|_L \leq 1\right\} < \infty.$$

Note that L' equipped with $\|\cdot\|_{L'}$ is a Banach function space satisfying the Fatou property (see [18, Theorem 1, § 68]).

For any elements f and g of L^0 such that fg is in L^1, we set

$$T_g(f) = \int_{\mathbb{R}^d} f(x)g(x)dx.$$

It is known that $g \mapsto T_g$ is an isometric linear map of L' into the topological dual space L^* of L (see [18, Theorem 2, § 69]). Therefore we shall look L' as a closed subspace of L^* by identifying any element g of L' with T_g.

3.2. **Banach lattices.** We recall the following definition from [10].

Definition 3.6. (1) A Banach space $(B, \|\cdot\|_B)$ is said to be a Banach lattice of functions (or an ideal Banach lattice) on $\mathbb{R}^d$ whenever it is a linear subspace of L^0 and satisfies the following property:

$$\left[\ f \in L^0, g \in B \text{ and } |f| \leq |g| \text{ in } L^0\ \right] \Longrightarrow \left[\ f \in B \text{ and } \|f\|_B \leq \|g\|_B\ \right].$$

(2) Let B be a Banach lattice of functions on $\mathbb{R}^d$.

(a) An element f of B is of absolutely continuous norm whenever

$\lim_{n\to\infty} \|f\chi_{E_n}\|_B = 0$ for any nonincreasing sequence $(E_n)_{n\geq 1}$ of measurable subsets of $\mathbb{R}^d$ such that $\left|\bigcap_{n\geq 1} E_n\right| = 0$.

(b) B is said to have an absolutely continuous norm whenever every element of B is of absolutely continuous norm.

Remark 3.7. (1) It is easy to see that, every Banach function space on $\mathbb{R}^d$ is a Banach lattice of functions on $\mathbb{R}^d$, but the converse is not true.

(2) It is well known (see [18, Theorem 1, § 72]) that an element of a Banach function space on $\mathbb{R}^d$ is of absolutely continuous norm in the Banach lattice of functions sense (Definition 3.6) if and only if it is of absolutely continuous norm in the Banach function space sense (Definition 3.2).

3.3. **Calderón product and Calderón's complex interpolation functors.** Let us assume that Y_0 and Y_1 are two Banach lattices of functions on $\mathbb{R}^d$ and $0 < \theta < 1$. The Calderón product $Y_0^{1-\theta}Y_1^{\theta}$ of Y_0 and Y_1 is defined by

$$Y_0^{1-\theta}Y_1^{\theta} = \bigcup_{f_0\in Y_0,\, f_1\in Y_1} \left\{ f : \mathbb{R}^d \to \mathbb{C} : \ |f| \leq |f_0|^{1-\theta}|f_1|^{\theta} \text{ in } L^0 \right\},$$

and for any element f of $Y_0^{1-\theta}Y_1^{\theta}$,

$$\|f\|_{Y_0^{1-\theta}Y_1^{\theta}} := \inf\left\{ \|f_0\|_{Y_0}^{1-\theta}\|f_1\|_{Y_1}^{\theta} : f_0 \in Y_0, f_1 \in Y_1 \text{ and } |f| \leq |f_0|^{1-\theta}|f_1|^{\theta} \text{ in } L^0 \right\}.$$

It is known that $Y_0^{1-\theta}Y_1^{\theta}$ equipped with the norm $\|\cdot\|_{Y_0^{1-\theta}Y_1^{\theta}}$ is a Banach lattice of functions on $\mathbb{R}^d$.

We shall use the following well known results.

Proposition 3.8. (1) ([17]). *We have*

$$[Y_0, Y_1]_\theta = \overline{Y_0 \cap Y_1}^{Y_0^{1-\theta}Y_1^{\theta}}. \tag{3.1}$$

(2) ([10, Theorem IV.1.14,]). *If $Y_0^{1-\theta}Y_1^{\theta}$ has an absolutely continuous norm, then*

$$[Y_0, Y_1]_\theta = Y_0^{1-\theta}Y_1^{\theta}. \tag{3.2}$$

Remark 3.9. If (h_0, h_1) is an element of $Y_0 \times Y_1$ such that h_0 or h_1 is of absolutely continuous norm, then $|h_0|^{1-\theta}|h_1|^{\theta}$ is of absolutely continuous norm in $Y_0^{1-\theta}Y_1^{\theta}$. Therefore, if Y_0 or Y_1 has an absolutely continuous norm, then $Y_0^{1-\theta}Y_1^{\theta}$ has an absolutely continuous norm, and so (3.2) holds true (see the remark just after [10, Theorem IV.1.14]).

Let us recall the following result on the relation between the Calderón product of Banach function spaces and their Köthe dual spaces.

Proposition 3.10 ([14, Theorem 2.10])**.** *If Y_0 and Y_1 are Banach function spaces on $\mathbb{R}^d$ satisfying the Fatou property, then*

$$\left(Y_0^{1-\theta}Y_1^{\theta}\right)' = (Y_0')^{1-\theta}(Y_1')^{\theta}.$$

A particular case of [1, Theorem 4.5.1] reads as follows.

Proposition 3.11. *If $Y_0 \cap Y_1$ is dense in both Y_0 and Y_1 then*

$$([Y_0, Y_1]_\theta)^* = [Y_0^*, Y_1^*]^\theta .$$

3.4. $F(q,p,\alpha)$**,** $(L^q, l^p)^\alpha$**, and** $\mathcal{H}(q',p',\alpha')$ **spaces.** In this subsection, we assume that $1 \le q \le \alpha \le p \le \infty$ unless otherwise specified. We note that the norms (2.1) and (2.2) are very similar. In order to emphasize on this, let us adopt the following notations:

- $\mathbf{P}_r = \left\{I_k^r : \ k \in \mathbb{Z}^d\right\}, \qquad r \in (0, \infty)$,
- $\mathcal{P}_u = \{\mathbf{P}_r : \ r \in (0,\infty)\}$.

Note that, for any positive real number r, $\mathbf{P}_r$ belongs to $\mathcal{P}$ and therefore $\mathcal{P}_u$ is a subset of $\mathcal{P}$.
For any element $\mathbf{P} = \{Q_i : i \in \mathbf{I}\}$ of $\mathcal{P}$ and any element f of L^0, we set

$$\begin{aligned}\|f\|_{(\mathbf{P},q,p,\alpha)} &= \left\|\left\{|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f\chi_{Q_i}\|_q\right\}_{i\in\mathbf{I}}\right\|_{l^p}\\ &= \begin{cases} \left[\displaystyle\sum_{i\in\mathbf{I}}\left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f\chi_{Q_i}\|_q\right)^p\right]^{\frac{1}{p}} & \text{if } p<\infty,\\ \displaystyle\sup_{i\in\mathbf{I}}|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f\chi_{Q_i}\|_q & \text{if } p=\infty\ .\end{cases}\end{aligned}$$

It is easy to see that for any element f of L^0,

$$||f||_{q,p,\alpha} = \sup_{\mathbf{P}\in\mathcal{P}_u}\|f\|_{(\mathbf{P},q,p,\alpha)} = \sup_{r>0}\|f\|_{(\mathbf{P}_r,q,p,\alpha)}$$

and

$$\|f\|_{F(q,p,\alpha)} = \sup_{\mathbf{P}\in\mathcal{P}}\|f\|_{(\mathbf{P},q,p,\alpha)}\ .$$

Throughout the remainder of this subsection, X denotes the space $(L^q, l^p)^\alpha$ or the space $F(q,p,\alpha)$, and $\|\cdot\|_X$ is its norm. We shall now give some auxiliary results on the space X.

Proposition 3.12. *X is a Banach function space on $\mathbb{R}^d$ satisfying the Fatou property.*

Proof. If $\alpha \in \{q,p\}$, then $X = L^\alpha$, and so the result is well known (see [18]).
We suppose that $q < \alpha < p$.
(1) We already know that X is a Banach space.
(2) We take $\mathcal{S} = \mathcal{P}_u$ if $X = (L^q, l^p)^\alpha$ and $\mathcal{S} = \mathcal{P}$ if $X = F(q,p,\alpha)$.
(a) Let f and g be two elements of L^0 such that $|f| \le |g|$ and g belongs to X. For any element $\mathbf{P} = \{Q_i : i \in \mathbf{I}\}$ of $\mathcal{S}$, we have

$$|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f\chi_{Q_i}\|_q \le |Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|g\chi_{Q_i}\|_q, \qquad i \in \mathbf{I}.$$

Therefore

$$\|f\|_{(\mathbf{P},q,p,\alpha)} \le \|g\|_{(\mathbf{P},q,p,\alpha)}.$$

So

$$\sup_{\mathbf{P}\in\mathcal{S}}\|f\|_{(\mathbf{P},q,p,\alpha)} \leq \sup_{\mathbf{P}\in\mathcal{S}}\|g\|_{(\mathbf{P},q,p,\alpha)}.$$

Thus we obtain

$$\|f\|_X \leq \|g\|_X < \infty.$$

(b) Let $(f_n)_{n\geq 1}$ be a sequence of nonnegative elements of X such that $(f_n)_{n\geq 1}\uparrow f$ and $\sup\limits_{n\geq 1}\|f_n\|_X < \infty$. We have

$$f_n \leq f_{n+1} \leq f \quad \text{in } L^0, \quad n \geq 1.$$

Therefore $(\|f_n\|_X)_{n\geq 1}$ is a nondecreasing sequence and satisfies

$$\lim_{n\to\infty}\|f_n\|_X = \sup_{n\geq 1}\|f_n\|_X \leq \|f\|_X. \tag{3.3}$$

Let us consider a real number t such that $0 \leq t < \|f\|_X$ and an element $\mathbf{P} = \{Q_i : i \in \mathbf{I}\}$ of $\mathcal{S}$ satisfying $\|f\|_{(\mathbf{P},q,p,\alpha)} > t$. There exists a finite subset $\mathbf{J}$ of $\mathbf{I}$ such that

$$\begin{cases} \left[\sum\limits_{i\in\mathbf{J}}\left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f\chi_{Q_i}\|_q\right)^p\right]^{\frac{1}{p}} > t & \text{if } p < \infty, \\ \sup\limits_{i\in\mathbf{J}}|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f\chi_{Q_i}\|_q > t & \text{if } p = \infty\ . \end{cases}$$

Let m be the number of elements of $\mathbf{J}$ and set

$$a = \begin{cases} \left[\sum\limits_{i\in\mathbf{J}}\left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f\chi_{Q_i}\|_q\right)^p - t^p\right]\dfrac{1}{m} & \text{if } p < \infty, \\ \sup\limits_{i\in\mathbf{J}}|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f\chi_{Q_i}\|_q - t & \text{if } p = \infty\ . \end{cases}$$

Since $(f_n)_{n\geq 1}\uparrow f$, by the monotone convergence theorem,

$$\left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f_n\chi_{Q_i}\|_q\right)\uparrow\left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f\chi_{Q_i}\|_q\right), \qquad i\in\mathbf{J},$$

and so there exists a positive integer n_a such that for any integer $n \geq n_a$,

$$\begin{cases} \left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f_n\chi_{Q_i}\|_q\right)^p > \left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f\chi_{Q_i}\|_q\right)^p - a & \text{if } p < \infty, \\ |Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f_n\chi_{Q_i}\|_q > |Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f\chi_{Q_i}\|_q - a & \text{if } p = \infty, \end{cases} \quad i \in \mathbf{J}.$$

Therefore, for any integer $n \geq n_a$,

$$\begin{cases} \sum\limits_{i\in\mathbf{J}}\left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f_n\chi_{Q_i}\|_q\right)^p > \sum\limits_{i\in\mathbf{J}}\left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f\chi_{Q_i}\|_q\right)^p - ma = t^p & \text{if } p < \infty, \\ \sup\limits_{i\in\mathbf{J}}|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f_n\chi_{Q_i}\|_q > \sup\limits_{i\in\mathbf{J}}|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f\chi_{Q_i}\|_q - a = t & \text{if } p = \infty. \end{cases}$$

Thus

$$\|f_n\|_{(\mathbf{P},q,p,\alpha)} > t, \qquad n \geq n_a.$$

This shows that

$$\|f_n\|_X = \sup_{\mathbf{P}\in\mathcal{S}} \|f_n\|_{(\mathbf{P},q,p,\alpha)} > t, \qquad n \geq n_a.$$

Therefore

$$\lim_{n\to\infty} \|f_n\|_X = \sup_{n\geq 1} \|f_n\|_X > t.$$

Since this is true for all $0 \leq t < \|f\|_X$, we get

$$\lim_{n\to\infty} \|f_n\|_X \geq \|f\|_X. \tag{3.4}$$

From (3.3) and (3.4) we deduce that $\lim_{n\to\infty} \|f_n\|_X = \|f\|_X$. □

Remark 3.13. From (2.3), Propositions 3.4, 3.5, and 3.12, Remark 3.7, we can easily deduce the following properties:

(1) X is a Banach lattice of functions on $\mathbb{R}^d$.
(2) X_a and $\ddot{X}$ are closed order ideals in X and therefore Banach lattices of functions on $\mathbb{R}^d$.
(3) X_a is included in $\ddot{X}$.
(4) if $1 \leq \alpha < \infty$, then X_a is equal to the closure in X of L^α, and therefore $X_a = \mathring{X}$.

As announced in Section 1, we shall now show that the spaces $\mathring{X}$, $\ddot{X}$, and X are different when $X = (L^q, l^p)^\alpha$. In order to do this, we need the following result.

Proposition 3.14. *$(L^q, l^\infty)^\alpha_a$ is included in $(L^q, \mathfrak{c}_0)$.*

Proof. Let f be an element of $(L^q, l^\infty)^\alpha_a$, and set for any positive integer n, $E_n = \mathbb{R}^d \backslash Q(0, 2n)$. It is clear that $(E_n)_{n\geq 1} \downarrow \varnothing$. Hence $\lim_{n\to\infty} \|f\chi_{E_n}\|_{q,\infty,\alpha} = 0$; that is, for any positive real number ϵ, there is an integer $m_\epsilon \geq 1$ such that

$$\|f\chi_{E_n}\|_{q,\infty,\alpha} < \epsilon, \qquad n \geq m_\epsilon.$$

We also have

$$\|f\chi_{I_k^1}\|_q = \|f\chi_{E_n}\chi_{I_k^1}\|_q \leq \|f\chi_{E_n}\|_{q,\infty,\alpha}\,, \qquad k \in \mathbb{Z}^d \text{ with } I_k^1 \subset E_n.$$

Therefore, for any positive real number ϵ,

$$\|f\chi_{I_k^1}\|_q < \epsilon, \qquad k \in \mathbb{Z}^d \text{ with } I_k^1 \subset E_{m_\epsilon}.$$

Thus f belongs to $(L^q, \mathfrak{c}_0)$. This ends the proof. □

Proposition 3.15. *Let us assume that $d = 1$, $1 \leq q < \alpha < p \leq \infty$ and $X = (L^q, l^p)^\alpha$. Then*

(1) *$\ddot{X}$ is strictly included in X when $p < \infty$,*
(2) *$\mathring{X}$ is strictly included in $\ddot{X}$ when $p = \infty$.*

Proof. (1) Suppose that $p < \infty$, and set $f(x) = x^{-\frac{1}{\alpha}}\chi_{(0,1)}(x)$.

(a) It is clear that $0 \leq f \leq e$, where $e(x) = x^{-\frac{1}{\alpha}}\chi_{\mathbb{R}_+^*}(x)$. We proved in [4] that e belongs to $X = (L^q, l^p)^\alpha$. Therefore, since X is a normed Köthe space (see Proposition 3.12), we can deduce that f belongs to X.

(b) We set, for any integer $n \geq 2$,

$$f_n = f\chi_{\mathbb{R}\setminus\left\{\frac{1}{n}<|f|<n\right\}}.$$

We have

$$\frac{1}{n} < |f(x)| < n \iff \left[\ \frac{1}{n} < x^{-\frac{1}{\alpha}} < n \ \text{ and } 0 < x < 1\ \right] \iff n^{-\alpha} < x < 1,$$

and so $f_n = f\chi_{E_n}$ with $E_n = (0\,, n^{-\alpha}]$.

Let us consider an element r of E_n and denote by $k_{n,r}$ the unique positive integer satisfying

$$k_{n,r} \leq \frac{1}{rn^{\alpha}} < k_{n,r} + 1.$$

We have

$$\begin{aligned}
{}_r\|f_n\|_{q,p} &= \left[\sum_{k\in\mathbb{Z}}\left(\int_{E_n\cap I_k^r} x^{-\frac{q}{\alpha}}dx\right)^{\frac{p}{q}}\right]^{\frac{1}{p}} \\
&= \left[\sum_{k=0}^{k_{n,r}-1}\left(\int_{kr}^{(k+1)r} x^{-\frac{q}{\alpha}}dx\right)^{\frac{p}{q}} + \left(\int_{k_{n,r}r}^{\frac{1}{n^{\alpha}}} x^{-\frac{q}{\alpha}}dx\right)^{\frac{p}{q}}\right]^{\frac{1}{p}} \\
&= \left(\frac{\alpha}{\alpha-q}\,r^{\frac{\alpha-q}{\alpha}}\right)^{\frac{1}{q}} \\
&\quad\times\left[\sum_{k=0}^{k_{n,r}-1}\left((k+1)^{\frac{\alpha-q}{\alpha}} - k^{\frac{\alpha-q}{\alpha}}\right)^{\frac{p}{q}} + \left(\left(\frac{1}{rn^{\alpha}}\right)^{\frac{\alpha-q}{\alpha}} - k_{n,r}^{\frac{\alpha-q}{\alpha}}\right)^{\frac{p}{q}}\right]^{\frac{1}{p}}.
\end{aligned}$$

Therefore

$$r^{\frac{1}{\alpha}-\frac{1}{q}}\,{}_r\|f_n\|_{q,p} > \left(\frac{\alpha}{\alpha-q}\right)^{\frac{1}{q}}.$$

Thus

$$\|f_n\|_{q,p,\alpha} > \left(\frac{\alpha}{\alpha-q}\right)^{\frac{1}{q}}.$$

This shows that f is not in $\ddot{X} = (L^q, \ddot{}\, l^p)^{\alpha}$.

(2) Suppose $p = \infty$, and set $E = \bigcup\limits_{m\geq 1} E_m$ with

$$E_m = \left(m - 1 + m^{\frac{\alpha}{\alpha-q}}, m + m^{\frac{\alpha}{\alpha-q}}\right), \qquad m \geq 1.$$

It has been stated (without detailed proof) in [13] that χ_E belongs to $X = (L^q, l^{\infty})^{\alpha}$ but does not belong to $\mathring{X}$. We shall prove that actually it is in $\ddot{X}\setminus\mathring{X}$.

(a) Let $J = (a, a+r)$ be an interval of length r. Then; First case: $r > 2^{\frac{\alpha}{\alpha-q}}$.

Since the distance $d_m = (m+1)^{\frac{\alpha}{\alpha-q}} - m^{\frac{\alpha}{\alpha-q}}$ between E_m and E_{m+1} increases with m, we have

$$|E \cap J| \leq |E \cap (1, 1+r)| \leq m_r,$$

where m_r stands for the greatest integer satisfying $m_r - 1 + m_r^{\frac{\alpha}{\alpha-q}} < r+1$. Since $m_r - 1 \geq 1$, we have $m_r^{\frac{\alpha}{\alpha-q}} < r$ and so, $m_r < \frac{\alpha-q}{\alpha}\ln(r)$. Therefore

$$r^{\frac{1}{\alpha}-\frac{1}{q}}\|\chi_E\chi_J\|_q \leq r^{\frac{1}{\alpha}-\frac{1}{q}}\left[\frac{\alpha-q}{\alpha}\ln(r)\right]^{\frac{1}{q}} = \left(\frac{\alpha-q}{\alpha}\right)^{\frac{1}{q}}\left[r^{\frac{q}{\alpha}-1}\ln(r)\right]^{\frac{1}{q}} \leq e^{-\frac{1}{q}}.$$

<u>Second case</u>: $r \leq 2^{\frac{\alpha}{\alpha-q}}$.

It is easy to see that

$$r^{\frac{1}{\alpha}-\frac{1}{q}}\|\chi_E\chi_J\|_q \leq r^{\frac{1}{\alpha}} \leq 2^{\frac{1}{\alpha-q}}.$$

From what proceeds we deduce that χ_E is in X, and therefore by point 3) of Proposition 3.4, it belongs to $\ddot{X}$.

(b) We have for any positive integer k, $\|\chi_E\chi_{E_k}\|_q = 1$, and so χ_E does not belong to $(L^q, \mathfrak{c}_0)$. Therefore, since $X_a = (L^q, l^\infty)_a^\alpha$ is included in $(L^q, \mathfrak{c}_0)$ (see Proposition 3.14), χ_E is not in X_a, which is equal to $\mathring{X}$ (see Remark 3.13). This ends the proof. □

Remark 3.16. Let us assume that $1 \leq q_j \leq \alpha_j \leq p_j \leq \infty$ for $j \in \{0,1\}$, $\theta \in (0,1)$ and that (X_0, X_1) is equal to either $\Big((L^{q_0}, l^{p_0})^{\alpha_0}, (L^{q_1}, l^{p_1})^{\alpha_1}\Big)$ or $\Big(F(q_0,p_0,\alpha_0), F(q_1,p_1,\alpha_1)\Big)$.

(1) Remark 3.13 asserts that for $j \in \{0,1\}$, the spaces X_j and $(X_j)_a$ are Banach lattices of functions on $\mathbb{R}^d$. Moreover, the spaces $(X_j)_a$ $(j=0,1)$ have an absolutely continuous norm. Consequently, from Remark 3.9, we deduce what follows:

$$[(X_0)_a\,,(X_1)_a]_\theta = (X_0)_a^{1-\theta}(X_1)_a^\theta,$$

$$[(X_0)_a\,,X_1]_\theta = (X_0)_a^{1-\theta}(X_1)^\theta,$$

$$[X_0\,,(X_1)_a]_\theta = X_0^{1-\theta}(X_1)_a^\theta.$$

(2) Proposition 3.12 asserts that the spaces X_j $(j=0,1)$ are Banach function spaces on $\mathbb{R}^d$ satisfying the Fatou property. Therefore, by Proposition 3.10 we have

$$\left(X_0^{1-\theta}X_1^\theta\right)' = (X_0')^{1-\theta}(X_1')^\theta\,. \tag{3.5}$$

We recall below some basic properties of $\mathcal{H}(q',p',\alpha')$.

Proposition 3.17 ([4]). (1) *$\mathcal{H}(q',p',\alpha')$, equipped with $\|\cdot\|_{\mathcal{H}(q',p',\alpha')}$, is a Banach function space on $\mathbb{R}^d$ in which $(L^{q'}, l^{p'})$ (and also $\mathcal{C}_c^\infty$ if $1<q$) is dense.*

(2) *If $1<q$, then $\mathcal{H}(q',p',\alpha')$ has an absolutely continuous norm and*

$$[\mathcal{H}(q',p',\alpha')]^* = [\mathcal{H}(q',p',\alpha')]' = (L^q, l^p)^\alpha. \tag{3.6}$$

(3) *If* $1 < q \leq \alpha < p \leq \infty$, *then* $\mathcal{H}(q', p', \alpha')$ *satisfies the Fatou property,*

$$[(L^q, l^p)^\alpha]' = \mathcal{H}(q', p', \alpha'), \tag{3.7}$$

and

$$[(L^q, l^p)^\alpha_a]^* = \mathcal{H}(q', p', \alpha'). \tag{3.8}$$

Remark 3.18. Let us assume that $1 < q_j \leq \alpha_j \leq p_j \leq \infty$ for j in $\{0, 1\}$ and that $\theta \in (0, 1)$. It follows from Point 2 of Proposition 3.17 and Remark 3.9 that

$$[\mathcal{H}(q'_0, p'_0, \alpha'_0), \mathcal{H}(q'_1, p'_1, \alpha'_1)]_\theta = \mathcal{H}(q'_0, p'_0, \alpha'_0)^{1-\theta}\mathcal{H}(q'_1, p'_1, \alpha'_1)^\theta.$$

4. Proofs of main results

Throughout this section, we assume, unless otherwise specified, that

- $1 \leq q_j \leq \alpha_j \leq p_j \leq \infty$ with $\alpha_j < \infty$ for j in $\{0, 1\}$,
- $0 < \theta < 1$, $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$, $\frac{1}{\alpha} = \frac{1-\theta}{\alpha_0} + \frac{\theta}{\alpha_1}$, and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$,
- $\Big(X_0,\ X_1,\ X\Big)$ is equal to $\Big((L^{q_0}, l^{p_0})^{\alpha_0},\ (L^{q_1}, l^{p_1})^{\alpha_1},\ (L^q, l^p)^\alpha\Big)$ or $\Big(F(q_0, p_0, \alpha_0),\ F(q_1, p_1, \alpha_1),\ F(q, p, \alpha)\Big)$,
- $\Big(\|\cdot\|_{X_0},\ \|\cdot\|_{X_1},\ \|\cdot\|_X\Big)$ is equal to $\Big(\|\cdot\|_{q_0,p_0,\alpha_0},\ \|\cdot\|_{q_1,p_1,\alpha_1},\ \|\cdot\|_{q,p,\alpha}\Big)$ or $\Big(\|\cdot\|_{F(q_0,p_0,\alpha_0)},\ \|\cdot\|_{F(q_1,p_1,\alpha_1)},\ \|\cdot\|_{F(q,p,\alpha)}\Big)$,
- $\mathcal{S} = \mathcal{P}_u$ if $\Big(X_0, X_1, X\Big)$ is equal to $\Big((L^{q_0}, l^{p_0})^{\alpha_0}, (L^{q_1}, l^{p_1})^{\alpha_1}, (L^q, l^p)^\alpha\Big)$,
- $\mathcal{S} = \mathcal{P}$ if $\Big(X_0, X_1, X\Big)$ is equal to $\Big(F(q_0, p_0, \alpha_0),\ F(q_1, p_1, \alpha_1),\ F(q, p, \alpha)\Big)$.

We need the following two lemmas for our proofs.

Lemma 4.1. *For any elements f, f_0 and f_1 of L^0 satisfying $|f| \leq |f_0|^{1-\theta}|f_1|^\theta$, it holds*

$$\|f\|_X \leq \|f_0\|_{X_0}^{1-\theta}\ \|f_1\|_{X_1}^\theta.$$

Therefore

$$X_0^{1-\theta}X_1^\theta \subset X.$$

Proof. For any element $\mathbf{P} = \{Q_i : i \in \mathbf{I}\}$ of $\mathcal{S}$, we have

$$|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\left\|f\chi_{Q_i}\right\|_q \leq \left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\left\|f_0\chi_{Q_i}\right\|_{q_0}\right)^{1-\theta}\left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\left\|f_1\chi_{Q_i}\right\|_{q_1}\right)^\theta, \quad i \in \mathbf{I}.$$

Therefore

$$\begin{aligned}\|f\|_{(\mathbf{P},q,p,\alpha)} &= \left\|\left\{|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\left\|f\chi_{Q_i}\right\|_q\right\}_{i\in\mathbf{I}}\right\|_{l^p} \\ &\leq \left\|\left\{\left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\left\|f_0\chi_{Q_i}\right\|_{q_0}\right)^{1-\theta}\left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\left\|f_1\chi_{Q_i}\right\|_{q_1}\right)^\theta\right\}_{i\in\mathbf{I}}\right\|_{l^p}.\end{aligned}$$

By Hölder's inequality, we get

$$\begin{aligned}\|f\|_{(\mathbf{P},q,p,\alpha)} &\le \left\|\left\{|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f_0\chi_{Q_i}\|_{q_0}\right\}_{i\in\mathbf{I}}\right\|_{l^{p_0}}^{1-\theta}\left\|\left\{|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f_1\chi_{Q_i}\|_{q_1}\right\}_{i\in\mathbf{I}}\right\|_{l^{p_1}}^{\theta}\\ &\le \|f_0\|_{(\mathbf{P},q_0,p_0,\alpha_0)}^{1-\theta}\|f_1\|_{(\mathbf{P},q_1,p_1,\alpha_1)}^{\theta}.\end{aligned}$$

This implies that

$$\sup_{\mathbf{P}\in\mathcal{S}}\|f\|_{(\mathbf{P},q,p,\alpha)} \le \sup_{\mathbf{P}\in\mathcal{S}}\|f_0\|_{(\mathbf{P},q_0,p_0,\alpha_0)}^{1-\theta}\sup_{\mathbf{P}\in\mathcal{S}}\|f_1\|_{(\mathbf{P},q_1,p_1,\alpha_1)}^{\theta},$$

and so

$$\|f\|_X \le \|f_0\|_{X_0}^{1-\theta}\|f_1\|_{X_1}^{\theta}.$$

Thus

$$X_0^{1-\theta}X_1^{\theta}\subset X.$$

□

Lemma 4.2. *Let us assume that* $\frac{\alpha_0}{\alpha_1}=\frac{q_0}{q_1}\le\frac{p_0}{p_1}$ *for* $j\in\{0,1\}$. *Then*

(1) *for* f *in* L^0 *and* j *in* $\{0,1\}$

$$\left\||f|^{\frac{q}{q_j}}\right\|_{X_j}\le\|f\|_X^{\frac{q}{q_j}}$$

and

$$\left\||f|^{\frac{q}{q_j}}\right\|_{X_j}=\|f\|_X^{\frac{q}{q_j}}\qquad \textit{if}\quad \frac{\alpha_0}{\alpha_1}=\frac{q_0}{q_1}=\frac{p_0}{p_1},$$

(2)

$$X_0^{1-\theta}X_1^{\theta}=X.$$

Proof. (1) Let j be in $\{0,1\}$ and let f belong to L^0. For any $\mathbf{P}=\{Q_i : i\in\mathbf{I}\}$ in $\mathcal{S}$, we have

$$|Q_i|^{\frac{1}{\alpha_j}-\frac{1}{q_j}}\left\||f|^{\frac{q}{q_j}}\chi_{Q_i}\right\|_{q_j}=\left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f\chi_{Q_i}\|_q\right)^{\frac{q}{q_j}},\qquad i\in\mathbf{I}.$$

Therefore

$$\left\|\left\{|Q_i|^{\frac{1}{\alpha_j}-\frac{1}{q_j}}\left\||f|^{\frac{q}{q_j}}\chi_{Q_i}\right\|_{q_j}\right\}_{i\in\mathbf{I}}\right\|_{l^{p_j}}=\left\|\left\{\left(|Q_i|^{\frac{1}{\alpha}-\frac{1}{q}}\|f\chi_{Q_i}\|_q\right)^{\frac{q}{q_j}}\right\}_{i\in\mathbf{I}}\right\|_{l^{p_j}}.$$

Note that

$$\begin{cases}\frac{\alpha_0}{\alpha_1}=\frac{q_0}{q_1}\le\frac{p_0}{p_1}\Longrightarrow\left[\frac{\alpha_j}{\alpha}=\frac{q_j}{q}\le\frac{p_j}{p},\quad j\in\{0,1\}\right],\\[2ex] \frac{\alpha_0}{\alpha_1}=\frac{q_0}{q_1}=\frac{p_0}{p_1}\Longrightarrow\left[\frac{\alpha_j}{\alpha}=\frac{q_j}{q}=\frac{p_j}{p},\quad j\in\{0,1\}\right].\end{cases}$$

Thus we get

$$\begin{cases} \left\| \left\{ |Q_i|^{\frac{1}{\alpha_j}-\frac{1}{q_j}} \left\| |f|^{\frac{q}{q_j}} \chi_{Q_i} \right\|_{q_j} \right\}_{i\in\mathbf{I}} \right\|_{l^{p_j}} \leq \left\| \left\{ |Q_i|^{\frac{1}{\alpha}-\frac{1}{q}} \left\| f\chi_{Q_i} \right\|_q \right\}_{i\in\mathbf{I}} \right\|_{l^p}^{\frac{q}{q_j}}, \\ \text{and if } \frac{\alpha_0}{\alpha_1} = \frac{q_0}{q_1} = \frac{p_0}{p_1}, \\ \left\| \left\{ |Q_i|^{\frac{1}{\alpha_j}-\frac{1}{q_j}} \left\| |f|^{\frac{q}{q_j}} \chi_{Q_i} \right\|_{q_j} \right\}_{i\in\mathbf{I}} \right\|_{l^{p_j}} = \left\| \left\{ |Q_i|^{\frac{1}{\alpha}-\frac{1}{q}} \left\| f\chi_{Q_i} \right\|_q \right\}_{i\in\mathbf{I}} \right\|_{l^p}^{\frac{q}{q_j}}. \end{cases}$$

This shows that

$$\begin{cases} \sup_{\mathbf{P}\in\mathcal{S}} \left\| |f|^{\frac{q}{q_j}} \right\|_{(\mathbf{P},q_j,p_j,\alpha_j)} \leq \sup_{\mathbf{P}\in\mathcal{S}} \|f\|_{(\mathbf{P},q,p,\alpha)}^{\frac{q}{q_j}}, \\ \sup_{\mathbf{P}\in\mathcal{S}} \left\| |f|^{\frac{q}{q_j}} \right\|_{(\mathbf{P},q_j,p_j,\alpha_j)} = \sup_{\mathbf{P}\in\mathcal{S}} \|f\|_{(\mathbf{P},q,p,\alpha)}^{\frac{q}{q_j}} \qquad \text{if } \frac{\alpha_0}{\alpha_1} = \frac{q_0}{q_1} = \frac{p_0}{p_1}. \end{cases}$$

and so

$$\begin{cases} \left\| |f|^{\frac{q}{q_j}} \right\|_{X_j} \leq \|f\|_X^{\frac{q}{q_j}}, \\ \left\| |f|^{\frac{q}{q_j}} \right\|_{X_j} = \|f\|_X^{\frac{q}{q_j}} \qquad \text{if } \frac{\alpha_0}{\alpha_1} = \frac{q_0}{q_1} = \frac{p_0}{p_1}. \end{cases}$$

(2) Let f be in X. By the result obtained in point (1), for j in $\{0,1\}$, $f_j = |f|^{\frac{q}{q_j}}$ belongs to X_j. Furthermore, $|f_0|^{1-\theta}|f_1|^{\theta} = |f|$. Consequently, f is in $X_0^{1-\theta}X_1^{\theta}$. This shows that

$$X \subset X_0^{1-\theta}X_1^{\theta}.$$

From this inclusion and Lemma 4.1, we deduce the claim. □

Following essentially the argumentation used by Hakim [9] and using the above results, we can give the proof of our first main result.

Proof of Theorem 1.1. Point (1) of Remark 3.13 asserts that the spaces $X_j (j = 0,1)$ are Banach lattices of functions on $\mathbb{R}^d$. Consequently, by (3.1) and Lemma 4.2, we have

$$[X_0, X_1]_\theta = \overline{X_0 \cap X_1}^X. \tag{4.1}$$

(a) Without loss of generality, we assume that $q_0 < q_1$.
Let $f \in X_0 \cap X_1$, $\mathbf{P} = \{Q_i : i \in \mathbf{I}\} \in \mathcal{S}$ and let n be a positive integer. We have

for any $i \in \mathbf{I}$,

$$\begin{aligned}
&\left\| |Q_i|^{\frac{1}{\alpha}-\frac{1}{q}} \left(f\chi_{\mathbb{R}^d \setminus \{\frac{1}{n}<|f|<n\}} \right) \chi_{Q_i} \right\|_q \\
&\le \left\| |Q_i|^{\frac{1}{\alpha}-\frac{1}{q}} |f|^{1-\frac{q_0}{q}} |f|^{\frac{q_0}{q}} \chi_{\{|f|\le\frac{1}{n}\}\cap Q_i} \right\|_q + \left\| |Q_i|^{\frac{1}{\alpha}-\frac{1}{q}} |f|^{1-\frac{q_1}{q}} |f|^{\frac{q_1}{q}} \chi_{\{|f|\ge n\}\cap Q_i} \right\|_q \\
&\le n^{\frac{q_0-q}{q}} |Q_i|^{\frac{1}{\alpha}-\frac{1}{q}} \left\| |f|^{\frac{q_0}{q}} \chi_{Q_i} \right\|_q + n^{\frac{q-q_1}{q}} |Q_i|^{\frac{1}{\alpha}-\frac{1}{q}} \left\| |f|^{\frac{q_1}{q}} \chi_{Q_i} \right\|_q \\
&= n^{\frac{q_0-q}{q}} |Q_i|^{\frac{1}{\alpha}-\frac{1}{q}} \| f\chi_{Q_i} \|_{q_0}^{\frac{q_0}{q}} + n^{\frac{q-q_1}{q}} |Q_i|^{\frac{1}{\alpha}-\frac{1}{q}} \| f\chi_{Q_i} \|_{q_1}^{\frac{q_1}{q}} \\
&= n^{\frac{q_0-q}{q}} \left(|Q_i|^{\frac{1}{\alpha_0}-\frac{1}{q_0}} \| f\chi_{Q_i} \|_{q_0} \right)^{\frac{q_0}{q}} + n^{\frac{q-q_1}{q}} \left(|Q_i|^{\frac{1}{\alpha_1}-\frac{1}{q_1}} \| f\chi_{Q_i} \|_{q_1} \right)^{\frac{q_1}{q}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\| \left\{ \left\| |Q_i|^{\frac{1}{\alpha}-\frac{1}{q}} \left(f\chi_{\mathbb{R}^d \setminus \{\frac{1}{n}<|f|<n\}} \right) \chi_{Q_i} \right\|_q \right\}_{i\in\mathbf{I}} \right\|_{l^p} \\
&\le n^{\frac{q_0-q}{q}} \left\| \left\{ \left(|Q_i|^{\frac{1}{\alpha_0}-\frac{1}{q_0}} \| f\chi_{Q_i} \|_{q_0} \right)^{\frac{q_0}{q}} \right\}_{i\in\mathbf{I}} \right\|_{l^p} \\
&\quad + n^{\frac{q-q_1}{q}} \left\| \left\{ \left(|Q_i|^{\frac{1}{\alpha_1}-\frac{1}{q_1}} \| f\chi_{Q_i} \|_{q_1} \right)^{\frac{q_1}{q}} \right\}_{i\in\mathbf{I}} \right\|_{l^p} \\
&= n^{\frac{q_0-q}{q}} \left\| \left\{ |Q_i|^{\frac{1}{\alpha_0}-\frac{1}{q_0}} \| f\chi_{Q_i} \|_{q_0} \right\}_{i\in\mathbf{I}} \right\|_{l^{p_0}}^{\frac{q_0}{q}} + n^{\frac{q-q_1}{q}} \left\| \left\{ |Q_i|^{\frac{1}{\alpha_1}-\frac{1}{q_1}} \| f\chi_{Q_i} \|_{q_1} \right\}_{i\in\mathbf{I}} \right\|_{l^{p_1}}^{\frac{q_1}{q}}.
\end{aligned}$$

(because of $\frac{pq_j}{q} = p_j$)

This shows that

$$\sup_{\mathbf{P}\in\mathcal{S}} \left\| f\chi_{\mathbb{R}^d \setminus \{\frac{1}{n}<|f|<n\}} \right\|_{(\mathbf{P},q,p,\alpha)} \le n^{\frac{q_0-q}{q}} \sup_{\mathbf{P}\in\mathcal{S}} \|f\|_{(\mathbf{P},q_0,p_0,\alpha_0)}^{\frac{q_0}{q}} + n^{\frac{q-q_1}{q}} \sup_{\mathbf{P}\in\mathcal{S}} \|f\|_{(\mathbf{P},q_1,p_1,\alpha_1)}^{\frac{q_1}{q}},$$

and so

$$\left\| f\chi_{\mathbb{R}^d \setminus \{\frac{1}{n}<|f|<n\}} \right\|_X \le n^{\frac{q_0-q}{q}} \|f\|_{X_0}^{\frac{q_0}{q}} + n^{\frac{q-q_1}{q}} \|f\|_{X_1}^{\frac{q_1}{q}}.$$

Since $q_0 < q_1$, we have $q_0 < q < q_1$. Therefore

$$\lim_{n\to\infty} \left\| f\chi_{\mathbb{R}^d \setminus \{\frac{1}{n}<|f|<n\}} \right\|_X = 0.$$

Thus f belongs to $\ddot{X}$. Hence we obtain $X_0 \cap X_1 \subset \ddot{X}$ and since $\ddot{X}$ is closed in X, we get

$$\overline{X_0 \cap X_1}^X \subset \ddot{X}. \tag{4.2}$$

(b) Let f be an element of $\ddot{X}$ and let n be a positive integer. We have, by Lemma 4.2,

$$\begin{aligned}
\left\| f\chi_{\{\frac{1}{n}<|f|<n\}} \right\|_{X_0} &\le \left\| \, |f|^{1-\frac{q}{q_0}} |f|^{\frac{q}{q_0}} \chi_{\{\frac{1}{n}<|f|\}} \right\|_{X_0} \le n^{\frac{q-q_0}{q}} \left\| \, |f|^{\frac{q}{q_0}} \right\|_{X_0} \\
&\le n^{\frac{q-q_0}{q}} \|f\|_X^{\frac{q}{q_0}} < \infty
\end{aligned}$$

and

$$\left\|f\chi_{\left\{\frac{1}{n}<|f|<n\right\}}\right\|_{X_1} \leq \left\| \, |f|^{1-\frac{q}{q_1}}|f|^{\frac{q}{q_1}}\chi_{\{|f|<n\}}\right\|_{X_1} \leq n^{\frac{q_1-q}{q}} \left\| \, |f|^{\frac{q}{q_1}}\right\|_{X_1}$$
$$\leq n^{\frac{q_1-q}{q}} \, \|f\|_X^{\frac{q}{q_1}} < \infty.$$

Therefore, $f\chi_{\left\{\frac{1}{n}<|f|<n\right\}}$ belongs to $X_0 \cap X_1$.
Since f belongs to $\ddot{X}$, $\left\{f\chi_{\left\{\frac{1}{n}<|f|<n\right\}}\right\}_{n\geq 1}$ converges to f in X and so f belongs to $\overline{X_0 \cap X_1}^X$. Thus

$$\ddot{X} \subset \overline{X_0 \cap X_1}^X. \tag{4.3}$$

The conjunction of (4.1), (4.2), and (4.3) shows that $[X_0, X_1]_\theta = \ddot{X}$. □

Proof of Theorem 1.2. (1) By Lemma 4.1, $(X_0)_a^{1-\theta}(X_1)_a^\theta$, $(X_0)_a^{1-\theta}X_1^\theta$ and $X_0^{1-\theta}(X_1)_a^\theta$ are continuously embedded in X. Furthermore, from Remark 3.9, the spaces $(X_0)_a^{1-\theta}(X_1)_a^\theta$, $(X_0)_a^{1-\theta}X_1^\theta$ and $X_0^{1-\theta}(X_1)_a^\theta$ are of absolutely continuous norm. Consequently, each of the three spaces $(X_0)_a^{1-\theta}(X_1)_a^\theta$, $(X_0)_a^{1-\theta}X_1^\theta$ and $X_0^{1-\theta}(X_1)_a^\theta$ is embedded in X_a.
(2) Let f be an element of X_a. We set $f_j = |f|^{\frac{q}{q_j}}$ for $j \in \{0,1\}$.
Let $\{E_n\}_{n\geq 1}$ be a nonincreasing sequence of measurable subsets of $\mathbb{R}^d$ such that $\left|\bigcap_{n\geq 1} E_n\right| = 0$. By Lemma 4.2 we have for any j in $\{0,1\}$,

$$\|f_j\chi_{E_n}\|_{X_j} \leq \|f\chi_{E_n}\|_X^{\frac{q}{q_j}}, \qquad n \geq 1.$$

Therefore, since f is of absolutely continuous norm, we get

$$\lim_{n\to\infty} \|f_j\chi_{E_n}\|_{X_j} = 0, \qquad j \in \{0,1\}.$$

Hence for j in $\{0,1\}$, f_j belongs to $(X_j)_a$, and so $|f| = |f_0|^{1-\theta}|f_1|^\theta$ is in all the three spaces $(X_0)_a^{1-\theta}(X_1)_a^\theta$, $(X_0)_a^{1-\theta}X_1^\theta$ and $X_0^{1-\theta}(X_1)_a^\theta$. Consequently, f belongs to $(X_0)_a^{1-\theta}(X_1)_a^\theta$, $(X_0)_a^{1-\theta}X_1^\theta$ and $X_0^{1-\theta}(X_1)_a^\theta$. Thus X_a is embedded in each of the spaces $(X_0)_a^{1-\theta}(X_1)_a^\theta$, $(X_0)_a^{1-\theta}X_1^\theta$ and $X_0^{1-\theta}(X_1)_a^\theta$.
(3) From the results obtained in point 1) and point 2) we have

$$(X_0)_a^{1-\theta}(X_1)_a^\theta = (X_0)_a^{1-\theta}X_1^\theta = X_0^{1-\theta}(X_1)_a^\theta = X_a \, .$$

The above equalities and Remark 3.16 imply that

$$[(X_0)_a \, , (X_1)_a]_\theta = [(X_0)_a \, , X_1]_\theta = [X_0 \, , (X_1)_a]_\theta = X_a.$$

Furthermore, $X_a = \mathring{X}$ and $(X_j)_a = \mathring{X}_j$ for j in $\{0,1\}$ (see point 4) of Remark 3.13). Thus we obtain

$$\left[\mathring{X}_0 \, , \mathring{X}_1\right]_\theta = \left[\mathring{X}_0 \, , X_1\right]_\theta = \left[X_0 \, , \mathring{X}_1\right]_\theta = \mathring{X}.$$

□

Proof of Theorem 1.3. (1) We have

$$\begin{aligned}[\mathcal{H}(q_0',p_0',\alpha_0'),\mathcal{H}(q_1',p_1',\alpha_1')]_\theta &= \mathcal{H}(q_0',p_0',\alpha_0')^{1-\theta}\mathcal{H}(q_1',p_1',\alpha_1')^\theta \quad \text{(by Remark 3.18)}\\ &= \left([(L^{q_0},l^{p_0})^{\alpha_0}]'\right)^{1-\theta}\left([(L^{q_1},l^{p_1})^{\alpha_1}]'\right)^\theta \quad \text{(by (3.7))}\\ &= \left([(L^{q_0},l^{p_0})^{\alpha_0}]^{1-\theta}[(L^{q_1},l^{p_1})^{\alpha_1}]^\theta\right)' \quad \text{(by (3.5))}\\ &= ((L^q,l^p)^\alpha)' \quad \text{(by Lemma 4.2)}\\ &= \mathcal{H}(q',p',\alpha'.) \quad \text{(by (3.7))}\end{aligned}$$

(2) Since, for j in $\{0,1\}$, $(L^{q_j},\mathring{l}^{p_j})^{\alpha_j}$ is the closure in $(L^{q_j},l^{p_j})^{\alpha_j}$ of $\mathcal{C}_c^\infty$, it is easy to see that $(L^{q_0},\mathring{l}^{p_0})^{\alpha_0}\cap(L^{q_1},\mathring{l}^{p_1})^{\alpha_1}$ is dense in both $(L^{q_0},\mathring{l}^{p_0})^{\alpha_0}$ and $(L^{q_1},\mathring{l}^{p_1})^{\alpha_1}$. Therefore

$$\begin{aligned}[\mathcal{H}(q_0',p_0',\alpha_0'),\mathcal{H}(q_1',p_1',\alpha_1')]^\theta &= [((L^{q_0},l^{p_0})_a^{\alpha_0})^*,((L^{q_1},l^{p_1})_a^{\alpha_1})^*]^\theta \quad \text{(by (3.8))}\\ &= \left[\left((L^{q_0},\mathring{l}^{p_0})^{\alpha_0}\right)^*,\left((L^{q_1},\mathring{l}^{p_1})^{\alpha_1}\right)^*\right]^\theta \\ &\qquad \text{(by Remark 3.13)}\\ &= \left(\left[(L^{q_0},\mathring{l}^{p_0})^{\alpha_0},(L^{q_1},\mathring{l}^{p_1})^{\alpha_1}\right]_\theta\right)^* \\ &\qquad \text{(by Proposition 3.11)}\\ &= \left((L^q,\mathring{l}^p)^\alpha\right)^* \quad \text{(by Theorem 1.2)}\\ &= (\ (L^q,l^p)_a^\alpha\ )^* \quad \text{(by Remark 3.13)}\\ &= \mathcal{H}(q',p',\alpha')\ . \quad \text{(by (3.8))}\end{aligned}$$

□

Proof of Theorem 1.4. Since $\mathcal{C}_c^\infty$ is dense in $\mathcal{H}(q_j',p_j',\alpha_j')$ for j in $\{0,1\}$, it is easy to see that $\mathcal{H}(q_0',p_0',\alpha_0')\cap\mathcal{H}(q_1',p_1',\alpha_1')$ is dense in both $\mathcal{H}(q_0',p_0',\alpha_0')$ and $\mathcal{H}(q_1',p_1',\alpha_1')$. Therefore

$$\begin{aligned}[(L^{q_0},l^{p_0})^{\alpha_0},(L^{q_1},l^{p_1})^{\alpha_1}]^\theta &= [\mathcal{H}(q_0',p_0',\alpha_0')^*,\mathcal{H}(q_1',p_1',\alpha_1')^*]^\theta \quad \text{(by (3.6))}\\ &= ([\mathcal{H}(q_0',p_0',\alpha_0'),\mathcal{H}(q_1',p_1',\alpha_1')]_\theta)^* \\ &\qquad \text{(by Proposition 3.11)}\\ &= \mathcal{H}(q',p',\alpha')^* \quad \text{(by Theorem 1.3)}\\ &= (L^q,l^p)^\alpha. \quad \text{(by (3.6))}\end{aligned}$$

□

Acknowledgement. The authors thank the anonymous reviewer and the editor for their suggestions for improvement of the presentation of this article.

References

1. J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Grundlehren der Mathematischen Wissenschaften, No. 223, Springer-Verlag, Berlin-New York, 1976.
2. O. Blasco, A. Ruiz and L. Vega, *Non-interpolation in Morrey-Campanato and block spaces,* Ann. Sc. Norm. Super. Pisa Cl. Sci. **28** (1999) 31–40.

3. A.P. Calderón, *Intermediate spaces and interpolation, the complex method,* Studia Math. **24** (1964), no. 2, 113–190.
4. N. Diarra and I. Fofana, *On preduals and Köthe duals of some subspaces of Morrey spaces*, J. Math. Anal. Appl. **496** (2021), no. 2, 124842.
5. H.G. Feichtinger and J. Feuto, *Pre-dual of Fofana's spaces*, MDPI Mathematics **7** (2019), no. 6, Art. No. 528, 12 pp.
6. I. Fofana, *Etude d'une classe d'espaces de fonctions contenant les espaces de Lorentz*, (French) [Study of a class of function spaces containing Lorentz spaces] Afr. Mat. **1** (1988), no. 1, 29–50.
7. I. Fofana, *Espace de fonctions à moyenne fractionnaire intégrable*, Thèse Docteur d'état es-sciences, numéro d'ordre 224, Université de Cocody, Abidjan R.C.I, 1995.
8. I. Fofana, F.R. Faléa and B.A. Kpata , *A class of subspaces of Morrey spaces and norm inequalities on Riesz potential operators*, Afr. Mat. **26** (2015) 717–739.
9. D.I. Hakim, *Calderón complex interpolation of Morrey spaces,* J. Indones. Math. Soc. **26** (2020), no. 1, 137–164 .
10. S.G. Krein, J.I. Petunin and E.M. Semenov, *Interpolation of linear operators*, Translated from the Russian by J. Szűcs, Translations of Mathematical Monographs 54, American Mathematical Society, Providence, RI. 1982.
11. P.G. Lemarié-Rieusset, *Erratum to: Multipliers and Morrey spaces,* Potential Anal. **41** (2014), no. 4, 1359–1362.
12. C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43** (1938) 126–166.
13. Y. Sawano and H. Tanaka, *The Fatou property of block spaces*, J. Math. Sci. Univ. Tokyo **22** (2015) 663–683
14. A.R. Schep, *Products and factors of Banach function spaces*, Positivity **14** (2010) 301–319.
15. D. Yang, W. Yuan and C. Zhuo, *Complex interpolation on Besov-type and Triebel-Lizorkin-type spaces,* Anal. Appl. (Singap.) **11** (2013), no. 5, 1350021, 45 pp.
16. W. Yuan, *Complex interpolation for predual spaces of Morrey-type spaces,* Taiwanese J. Math. **18** (2014), no. 5, 1527–1548.
17. W. Yuan, *Complex interpolation and Calderon product of quasi-Banach spaces,* Front. Math. China **11** (2016), no. 3, 753–762.
18. A.C. Zaanen, *An Introduction to the Theory of Integration*, North-Holland Publication Co. Amsterdam, 1967.

[1]Laboratoire de Mathematiques et Applications, UFR Mathematiques et Informatique,, Universite Felix Houphouet Boigny, Abidjan-Cocody, 22 BP 582, Abidjan 22, Cote d'ivoire.

Email address: `nouffoud@yahoo.fr; fofana_ib_math_ab@yahoo.fr`