Khayyam J. Math. 10 (2024), no. 1, 1-9
DOI: 10.22034/KJM.2023.385509.2768

Khayyam Journal of Mathematics
emis.de/journals/KJM
kjm-math.org

CHARACTERIZATION OF JORDAN HOMOMORPHISMS AND JORDAN DERIVATIONS

ABBAS ZIVARI-KAZEMPOUR

Communicated by H.R. Ebrahimi Vishki

Abstract. We show that if $f : A \longrightarrow B$ is a continuous linear map between Banach algebras satisfying $f(a \circ b) = f(a) \circ f(b)$ for all $a, b \in A$ with $a \circ b = e_A$ or $ab = ba = e_A$, then f is a Jordan homomorphism. It is also proved that if $\delta : A \longrightarrow X$ is a continuous linear map satisfying $\delta(a \circ b) = \delta(a)b + a\delta(b)$ for all $a, b \in A$ with $a \circ b = w$, where $w \in Z(A)$ is a right (or left) separating point of Banach A-bimodule X, then δ is a generalized Jordan derivation.

1. Introduction and preliminaries

Let A be a unital Banach algebra with unit e_A and let X be a unital Banach A-bimodule. A linear map $\delta : A \longrightarrow X$ is called a *derivation* [respectively, *generalized derivation*] if for all $a, b \in A$,

$$\delta(ab) = \delta(a)b + a\delta(b), \qquad [\delta(ab) = \delta(a)b + a\delta(b) - a\delta(e_A)b],$$

and it is called a *Jordan derivation* [respectively, *generalized Jordan derivation*] if

$$\delta(a^2) = \delta(a) \bullet a, \quad [\delta(a^2) = \delta(a) \bullet a - a\delta(e_A)a], \quad a \in A,$$

where " $\bullet$ " denotes the Jordan product on X:

$$a \bullet x = x \bullet a = ax + xa, \quad a \in A, \ \ x \in X.$$

Obviously, δ is a Jordan derivation [generalized Jordan derivation] if and only if

$$\delta(a \circ b) = \delta(a) \bullet b + a \bullet \delta(b), \ \ [\delta(a \circ b) = \delta(a) \bullet b + a \bullet \delta(b) - a\delta(e_A)b - b\delta(e_A)a],$$

for all $a, b \in A$. Here " $\circ$ " denotes the Jordan product $a \circ b = ab + ba$ on A.

Date: Received: 13 February 2023; Revised: 3 November 2023; Accepted: 8 January 2024.
2020 *Mathematics Subject Classification.* Primary 47B47; Secondary 47B48.
Key words and phrases. Jordan homomorphism, (generalized) Jordan derivation, separating point.

It is clear that each derivation (respectively, generalized derivation) is a Jordan derivation (respectively, generalized Jordan derivation), but the converse is failed in general [5].

It is proved by Johnson [5, Theorem 6.3] that every continuous Jordan derivation from C^*-algebra A into any Banach A-bimodule X is a derivation.

Recently, several authors have studied the linear maps that satisfy the derivation equation whether $ab = 0$, or ab is a non-trivial idempotent. We refer the reader to [1, 2, 4, 6] for a full account of the topic and a list of references.

We say that $w \in A$ is a left (right) separating point of A-bimodule X if the condition $wx = 0$ $[xw = 0]$ for $x \in X$ implies that $x = 0$.

A linear map $f : A \longrightarrow B$ between two Banach algebras A and B is called Jordan homomorphism if $f(a \circ b) = f(a) \circ f(b)$ for all $a, b \in A$, which is equivalent to assuming that $f(a^2) = f(a)^2$ for all $a \in A$.

Some characterizations of Jordan homomorphisms on Banach algebras were obtained by the author in [8–10].

In this paper, we show that f is a Jordan homomorphism whenever

$$f(a \circ b) = f(a) \circ f(b),$$

for all $a, b \in A$ with $a \circ b = e_A$. Moreover, under special hypotheses, it is proved that f is a Jordan homomorphism if and only if

$$a, b \in A, \quad ab = ba = e_A \implies \quad f(a \circ b) = f(a) \circ f(b).$$

As a consequence we characterize [generalized] Jordan derivations on Banach algebras. We also investigate the continuous linear maps from a Banach algebra A into a Banach A-bimodule X satisfying

$$a, b \in A, \quad a \circ b = w \implies \quad \delta(a \circ b) = \delta(a)b + a\delta(b),$$

where $w \in Z(A)$ is a right or left separating point of X and $Z(A)$ is the center of A.

Lemma 1.1 ([7, Lemma 6.3.2]). *Let $f : A \longrightarrow B$ be a Jordan homomorphism. Then*

$$f(aba) = f(a)f(b)f(a), \quad a, b \in A.$$

Through this paper, A and B are two Banach algebras, where A is unital and X is a unital Banach A-bimodule, unless indicated otherwise.

2. Characterization of Jordan homomorphisms

We commence with the following result, which is our first main theorem.

Theorem 2.1. *Let $f : A \longrightarrow B$ be a continuous linear map such that*

$$f(a \circ b) = f(a) \circ f(b),$$

for all $a, b \in A$ with $a \circ b = e_A$. Then

$$f(e_A)f(a^2) + f(a^2)f(e_A) = 2f(a)^2, \quad a \in A.$$

Moreover, if for all $a \in A$,

$$f(a) = f(a)f(e_A) = f(e_A)f(a),$$

then f is a Jordan homomorphism.

Proof. Since $\frac{1}{2}(e_A \circ e_A) = e_A$, we get $f(e_A) = f(e_A)^2$.

Let $a \in A$ be arbitrary. For $\lambda \in \mathbb{C}$ with $|\lambda| < 1/\|a\|$, $e_A - \lambda a$ is invertible and $(e_A - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$. It is obvious that

$$\frac{1}{2}(e_A - \lambda a) \circ (e_A - \lambda a)^{-1} = e_A;$$

thus it follows from the continuity of f that

$$\begin{aligned}
2f(e_A) &= f(e_A - \lambda a) f\left(\sum_{n=0}^{\infty} \lambda^n a^n\right) + f\left(\sum_{n=0}^{\infty} \lambda^n a^n\right) f(e_A - \lambda a)\\
&= \big(f(e_A) - \lambda f(a)\big) \sum_{n=0}^{\infty} \lambda^n f(a^n) + \sum_{n=0}^{\infty} \lambda^n f(a^n) \big(f(e_A) - \lambda f(a)\big)\\
&= f(e_A) f(e_A) + f(e_A) \sum_{n=1}^{\infty} \lambda^n f(a^n) - \lambda f(a) \sum_{n=0}^{\infty} \lambda^n f(a^n)\\
&\quad + f(e_A) f(e_A) + \sum_{n=1}^{\infty} \lambda^n f(a^n) f(e_A) - \lambda \sum_{n=0}^{\infty} \lambda^n f(a^n) f(a)\\
&= f(e_A) \sum_{n=0}^{\infty} \lambda^{n+1} f(a^{n+1}) - \lambda f(a) \sum_{n=0}^{\infty} \lambda^n f(a^n)\\
&\quad + \sum_{n=0}^{\infty} \lambda^{n+1} f(a^{n+1}) f(e_A) - \lambda \sum_{n=0}^{\infty} \lambda^n f(a^n) f(a).
\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} \lambda^{n+1} [f(e_A) f(a^{n+1}) - f(a) f(a^n) + f(a^{n+1}) f(e_A) - f(a^n) f(a)] = 0,$$

for all $\lambda \in \mathbb{C}$, with $|\lambda| < 1/\|a\|$. Hence

$$f(e_A) f(a^{n+1}) + f(a^{n+1}) f(e_A) = f(a) f(a^n) + f(a^n) f(a),$$

for $n = 0, 1, 2, \ldots$. Taking $n = 1$, we obtain

$$f(e_A) f(a^2) + f(a^2) f(e_A) = 2f(a)^2, \qquad a \in A.$$

If $f(a) = f(a) f(e_A) = f(e_A) f(a)$ for all $a \in A$, then it follows that $f(a^2) = f(a)^2$, and hence f is a Jordan homomorphism. □

Proposition 2.2. *Let A and B be two unital Banach algebras and let $f : A \longrightarrow B$ be a unital continuous linear map such that $f(ab) = f(a)f(b)$ for all $a, b \in A$ with $ab = e_A$. Then f is a Jordan homomorphism.*

Proof. Since $e_A e_A = e_A$, we get $f(e_A)^2 = f(e_A)$. Let $a \in A$. For $\lambda \in \mathbb{C}$, with $|\lambda| < 1/\|a\|$, $e_A - \lambda a$ is invertible and $(e_A - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$. Noting that

$$(e_A - \lambda a)(e_A - \lambda a)^{-1} = e_A,$$

thus by our assumption and a similar argument of Theorem 2.1, we obtain

$$f(e_A)f(a^{n+1}) = f(a)f(a^n),$$

for $n = 0, 1, 2, \dots$ and for every $a \in A$. Taking $n = 1$ and using the fact that $f(e_A) = e_B$, we conclude that $f(a^2) = f(a)^2$ for all $a \in A$. □

The set of all invertible elements of A is denoted by $Inv(A)$.

Corollary 2.3. *Let A and B be two unital Banach algebras and let $f : A \longrightarrow B$ be a unital continuous linear map such that $f(aa^{-1}) = f(a)f(a^{-1})$ for all $a \in Inv(A)$. Then f is a Jordan homomorphism.*

Similar to the proof of Theorem 2.1, we can obtain the following result.

Theorem 2.4. *Let $f : A \longrightarrow B$ be a continuous linear map such that*

$$a, b \in A, \quad ab = ba = e_A \implies f(a \circ b) = f(a) \circ f(b).$$

If $f(a) = f(a)f(e_A) = f(e_A)f(a)$ for every $a \in A$, then f is a Jordan homomorphism.

Next, we show that the converse of Theorem 2.4 is also true with additional hypothesis.

Theorem 2.5. *Suppose that $f : A \longrightarrow B$ is a Jordan homomorphism. Then*

$$\big(f(ab) - f(a)f(b)\big)f(a) = 0,$$

for all $a, b \in A$ with $ab = e_A$.

Proof. Since f is a Jordan homomorphism, we have

$$f(x)f(e_A) = f(e_A)f(x),$$

for all $x \in A$, and hence

$$f(x) = f(x)f(e_A) = f(e_A)f(x), \quad x \in A. \tag{2.1}$$

Now, let $a, b \in A$ with $ab = e_A$. Then $a = aba$, and by Lemma 1.1, we have

$$f(a) = f(aba) = f(a)f(b)f(a). \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$\big(f(e_A) - f(a)f(b)\big)f(a) = 0,$$

for every $a, b \in A$ with $ab = e_A$. □

The next result is a consequence of Theorem 2.5.

Corollary 2.6. *Let A and B be two unital Banach algebras and let $f : A \longrightarrow B$ be a Jordan homomorphism. If $f(a) \in Inv(B)$ for all $a \in A$, then $f(ab) = f(a)f(b)$ for all $a, b \in A$ with $ab = e_A$.*

It should be pointed out that by the hypotheses of the corollary above, we get $f(a \circ b) = f(a) \circ f(b)$ for all $a, b \in A$ with $ab = ba = e_A$.

Let us mention an example of a Jordan homomorphism $f : A \longrightarrow B$, where the identity $f(a \circ b) = f(a) \circ f(b)$ for all $a, b \in A$ with $a \circ b = e_A$ does not imply that f is a homomorphism.

Example 2.7. Let

$$A = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{bmatrix} : \quad a_{11}, a_{12}, a_{22} \in \mathbb{C} \right\}.$$

We make $X = \mathbb{C}$ an A-bimodule by defining

$$a\lambda = a_{22}\lambda, \quad \lambda a = \lambda a_{11}, \quad \lambda \in \mathbb{C},\ a \in A.$$

Consider the linear map $\delta : A \longrightarrow X$ defined by $\delta(a) = a_{12}$. Note that $\delta(e_A) = 0$. Then $\delta(ab) = \delta(b)a + b\delta(a)$ for all $a, b \in A$ and hence δ is a Jordan derivation. However, δ is not a derivation. Take

$$B = \left\{ \begin{bmatrix} a & x \\ 0 & a \end{bmatrix} : \quad a \in A,\ x \in X \right\}.$$

Then B becomes a unital Banach algebra under the usual matrix operations. Define a linear map $f : A \longrightarrow B$ by

$$f(a) = \begin{bmatrix} a & \delta(a) \\ 0 & a \end{bmatrix}, \quad a \in A.$$

Then for all $a, b \in A$ with $a \circ b = e_A$, we have

$$f(a) \circ f(b) = \begin{bmatrix} a \circ b & \delta(a \circ b) \\ 0 & a \circ b \end{bmatrix} = \begin{bmatrix} e_A & 0 \\ 0 & e_A \end{bmatrix} = f(a \circ b).$$

Therefore f is a Jordan homomorphism by Theorem 2.1, but it is not a homomorphism.

3. Characterization of Jordan derivations

In this section, we characterize continuous linear maps on Banach algebras, which are necessarily [generalized] Jordan derivations.

Theorem 3.1. *Let $\delta : A \longrightarrow X$ be a continuous linear map.*

(1) *δ is a Jordan derivation if and only if $\delta(a \circ b) = \delta(a) \bullet b + a \bullet \delta(b)$ for all $a, b \in A$ with $a \circ b = e_A$.*

(2) *δ is a generalized Jordan derivation if and only if for every $a, b \in A$ with $a \circ b = e_A$,*

$$\delta(a \circ b) = \delta(a) \bullet b + a \bullet \delta(b) - a\delta(e_A)b - b\delta(e_A)a. \tag{3.1}$$

Proof. (1) Let $\delta(a \circ b) = \delta(a) \bullet b + a \bullet \delta(b)$ for all $a, b \in A$ with $a \circ b = e_A$. Let f and B be as in Example 2.7. Then $f(a \circ b) = f(a) \circ f(b)$ for all $a, b \in A$ with $a \circ b = e_A$. So f is a Jordan homomorphism by Theorem 2.1 and hence δ is a Jordan derivation. The converse is clear.

(2) Suppose that equality (3.1) holds for all $a, b \in A$ with $a \circ b = e_A$. Define a linear map $D : A \longrightarrow X$ by $D(a) = \delta(a) - a\delta(e_A)$. Then $D(a \circ b) = D(a) \bullet b + a \bullet D(b)$ for all $a, b \in A$ with $a \circ b = e_A$, and hence (1) implies that D is a Jordan derivation. This means that δ is a generalized Jordan derivation. □

Theorem 3.2. *Let $\delta : A \longrightarrow X$ be a continuous linear map.*

(1) *δ is a Jordan derivation if and only if*

$$a, b \in A, \quad ab = ba = e_A \implies \delta(a \circ b) = \delta(a) \bullet b + a \bullet \delta(b). \tag{3.2}$$

(2) *δ is a generalized Jordan derivation if and only if for all $a, b \in A$ with $ab = ba = e_A$,*

$$\delta(a \circ b) = \delta(a) \bullet b + a \bullet \delta(b) - a\delta(e_A)b - b\delta(e_A)a.$$

Proof. (1) If δ is a Jordan derivation, then (3.2) holds. For the converse, let f and B be as in Example 2.7. Then $f(a \circ b) = f(a) \circ f(b)$ for all $a, b \in A$ with $ab = ba = e_A$. Consequently, f is a Jordan homomorphism by Theorem 2.4, and hence δ is a Jordan derivation.

Part (2) can be proved by similar argument as in the part (2) of Theorem 3.1. □

An immediate but noteworthy result to Theorem 3.2 is the following result.

Corollary 3.3. *Let $\delta : A \longrightarrow X$ be a continuous linear map.*

(1) *[6, Corollary 2.3] δ is a Jordan derivation if and only if for all $a, b \in A$ with $ab = e_A$, $\delta(ab) = \delta(a)b + a\delta(b)$.*
(2) *δ is a generalized Jordan derivation if and only if for every $a, b \in A$ with $ab = e_A$, $\delta(ab) = \delta(a)b + a\delta(b) - a\delta(e_A)b$.*

Proof. Let for all $a, b \in A$ with $ab = e_A$,

$$\delta(ab) = \delta(a)b + a\delta(b). \tag{3.3}$$

Then

$$a, b \in A, \quad ba = e_A \implies \delta(ba) = \delta(b)a + b\delta(a). \tag{3.4}$$

By (3.3) and (3.4), $\delta(a \circ b) = \delta(a) \bullet b + a \bullet \delta(b)$ for all $a, b \in A$ with $ab = ba = e_A$. Thus, δ is a Jordan derivation by Theorem 3.2.

Conversely, suppose that δ is a Jordan derivation. Let f and B be as in Example 2.7. Since $\delta(e_A) = 0$, it follows that

$$f(a) = f(a)f(e_A) = f(e_A)f(a), \quad a \in A. \tag{3.5}$$

Now, let $a, b \in A$ with $ab = e_A$. As δ is a Jordan derivation, f is a Jordan homomorphism, and thus by Lemma 1.1, we have

$$f(a) = f(aba) = f(a)f(b)f(a), \tag{3.6}$$

for all $a, b \in A$ with $ab = e_A$. It follows from (3.5) and (3.6) that

$$f(e_A)f(a) = f(a)f(b)f(a),$$

which yields that $\big(\delta(a)b + a\delta(b)\big)a = 0$ for all $a, b \in A$ with $ab = e_A$. Multiplying this equality from the right by b, we reach the desired result.

(2) follows from (1). □

4. Generalized Jordan derivations

By Theorem 3.1, $\delta : A \longrightarrow X$ is a generalized Jordan derivation if and only if equality (3.1) holds for all $a, b \in A$ with $a \circ b = e_A$. However, the following result is another tool for characterizing generalized Jordan derivations.

Theorem 4.1. *Let $w \in Z(A)$ be a right or left separating point of X and let $\delta : A \longrightarrow X$ be a continuous linear map satisfying*

$$a, b \in A, \quad a \circ b = w \implies \quad \delta(a \circ b) = \delta(a)b + a\delta(b).$$

Then δ is a generalized Jordan derivation and

$$\delta(aw) = \delta(a)w + a\delta(w) - \delta(e_A)aw, \quad \delta(wa) = \delta(w)a + w\delta(a) - wa\delta(e_A).$$

Proof. Since $\frac{1}{2}(e_A \circ w) = w$, it follows that $\delta(w) = \delta(e_A)w = w\delta(e_A)$.

Let $a \in A$ be arbitrary. For $\lambda \in \mathbb{C}$, with $|\lambda| < 1/\|a\|$, $e_A - \lambda a$ is invertible and $(e_A - \lambda a)^{-1} = \sum_{n=0}^{\infty} \lambda^n a^n$. As

$$\frac{1}{2}(e_A - \lambda a) \circ (e_A - \lambda a)^{-1} w = w,$$

so from the continuity of δ, we have

$$\begin{aligned}
2\delta(w) &= \delta(e_A - \lambda a)(e_A - \lambda a)^{-1}w + (e_A - \lambda a)\delta\big((e_A - \lambda a)^{-1}w\big) \\
&= \delta(e_A - \lambda a)\sum_{n=0}^{\infty} \lambda^n a^n w + (e_A - \lambda a)\delta\left(\sum_{n=0}^{\infty} \lambda^n a^n w\right) \\
&= \delta(e_A - \lambda a)w + \delta(e_A - \lambda a)\sum_{n=1}^{\infty} \lambda^n a^n w \\
&\quad + (e_A - \lambda a)\delta(w) + (e_A - \lambda a)\sum_{n=0}^{\infty} \lambda^n \delta(a^n w) \\
&= 2\delta(w) - \lambda\delta(a)w - \lambda a\delta(w) \\
&\quad + \delta(e_A - \lambda a)\sum_{n=1}^{\infty} \lambda^n a^n w + (e_A - \lambda a)\sum_{n=1}^{\infty} \lambda^n \delta(a^n w) \\
&= 2\delta(w) + \delta(e_A)\sum_{n=1}^{\infty} \lambda^n a^n w - \lambda\delta(a)\sum_{n=0}^{\infty} \lambda^n a^n w \\
&\quad + \sum_{n=1}^{\infty} \lambda^n \delta(a^n w) - \lambda a\sum_{n=0}^{\infty} \lambda^n \delta(a^n w).
\end{aligned}$$

Therefore

$$\sum_{n=0}^{\infty} \lambda^{n+1}[\delta(e_A)a^{n+1}w - \delta(a)a^n w + \delta(a^{n+1}w) - a\delta(a^n w)] = 0,$$

for all $\lambda \in \mathbb{C}$, with $|\lambda| < 1/\|a\|$. Hence

$$\delta(a^{n+1}w) = \delta(a)a^n w + a\delta(a^n w) - \delta(e_A)a^{n+1}w,$$

for $n = 0, 1, 2, \ldots$. In particular, we have

$$\delta(aw) = \delta(a)w + a\delta(w) - \delta(e_A)aw \tag{4.1}$$

and

$$\delta(a^2 w) = \delta(a)aw + a\delta(aw) - \delta(e_A)a^2 w, \tag{4.2}$$

for all $a \in A$. It follows from (4.1) and (4.2) that

$$\delta(a^2w) = \delta(a)aw + a[\delta(a)w + a\delta(w) - \delta(e_A)aw] - \delta(e_A)a^2w. \tag{4.3}$$

Interchanging a by a^2 in (4.1), we get

$$\delta(a^2w) = \delta(a^2)w + a^2\delta(w) - \delta(e_A)a^2w. \tag{4.4}$$

Comparing (4.3) and (4.4), we arrive at

$$\delta(a^2)w = \delta(a)aw + a\delta(a)w - a\delta(e_A)aw.$$

If w is a right separating point of X, then $\delta(a^2) = \delta(a)a + a\delta(a) - a\delta(e_A)a$ for all $a \in A$. Consequently, δ is a generalized Jordan derivation. Similarly, by using the equality

$$w(e_A - \lambda a)^{-1} \circ \frac{1}{2}(e_A - \lambda a) = w,$$

we get

$$\delta(wa) = \delta(w)a + w\delta(a) - wa\delta(e_A)$$

and

$$\delta(wa^2) = \delta(wa)a + wa\delta(a) - wa^2\delta(e_A),$$

for all $a \in A$. Thus,

$$w\delta(a^2) = wa\delta(a) + w\delta(a)a - wa\delta(e_A)a.$$

If w is a left separating point of X, then $\delta(a^2) = a\delta(a) + \delta(a)a - a\delta(e_A)a$, and hence δ is a generalized Jordan derivation. □

Similar to the proof of Theorem 4.1, we can obtain the following result.

Proposition 4.2. *Let $w \in Z(A)$ be a right or left separating point of X and let $\delta : A \longrightarrow X$ be a continuous linear map satisfying*

$$a, b \in A, \quad ab = ba = w \implies \quad \delta(a \circ b) = \delta(a)b + a\delta(b).$$

Then δ is a generalized Jordan derivation.

In particular, if $w = e_A$, then $a\delta(e_A) = \delta(e_A)a$ for all $a \in A$

From Proposition 4.2, we have the following result.

Corollary 4.3. *Let $\delta : A \longrightarrow X$ be a continuous linear map such that for all $a \in Inv(A)$, $\delta(a \circ a^{-1}) = \delta(a)a^{-1} + a\delta(a^{-1})$. Then δ is a generalized Jordan derivation.*

The converse of Theorem 4.1 is not true in general. The following example illustrates this fact.

Example 4.4. Let

$$A = \left\{ \begin{bmatrix} s & t \\ 0 & r \end{bmatrix} : \quad s, t, r \in \mathbb{R} \right\}.$$

Define a linear map $\delta : A \longrightarrow A$ by

$$\delta(\begin{bmatrix} s & t \\ 0 & r \end{bmatrix}) = \begin{bmatrix} s & 0 \\ 0 & r \end{bmatrix}.$$

Then $\delta(ab) = \delta(a)b + a\delta(b) - a\delta(e_A)b$ for all $a, b \in A$. Certainly, δ is a generalized Jordan derivation, but the equality $\delta(a \circ b) = \delta(a)b + a\delta(b)$ is failed, in general, even for $a \circ b = e_A$. For example, take

$$a = \begin{bmatrix} \frac{-1}{2} & 1 \\ 0 & \frac{1}{2} \end{bmatrix}, \qquad b = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If $\delta(a) = a\delta(e_A)$ for all $a \in A$, then δ is a generalized Jordan derivation, but Example 4.4 shows that the converse is not true, in general. However, for commutative C^*-algebra it is holds according the next result.

Corollary 4.5. *Let A be a commutative C^*-algebra and let $\delta : A \longrightarrow X$ be a continuous linear map such that $\delta(a \circ a^{-1}) = \delta(a)a^{-1} + a\delta(a^{-1})$ for all $a \in Inv(A)$. Then $\delta(a) = a\delta(e_A)$ for all $a \in A$. In particular, δ is a generalized derivation.*

Proof. By Corollary 4.3, δ is a generalized Jordan derivation. Define a linear mapping $D : A \longrightarrow X$ by $D(a) = \delta(a) - a\delta(e_A)$ for all $a \in A$. Clearly, D is a Jordan derivation, and hence it is a derivation by Johnson's result. On account of [3, Theorem 2.8.63], D is identically zero. Consequently, $\delta(a) = a\delta(e_A)$ for all $a \in A$. ☐

Acknowledgement. The author gratefully acknowledges the helpful comments of the anonymous referees.

References

1. J. Alaminos, M. Bresar, J. Extremera and A.R. Villena, *Maps preserving zero products*, Studia Math. **193** (2009) 131–159.
2. G. An and J. Li, *Characterization of linear mappings through zero products or zero Jordan products* Electron. J. Linear Algebra **31** (2016) 408–424.
3. H.G. Dales, *Banach algebras and automatic continuity*, The Clarendon Press, Oxford University Press, New York, 2000.
4. H. Ghahramani, *On derivations and Jordan derivations through zero products*, Oper. Matrices, **8** (2014) 759–771.
5. B.E. Johnson, *Symmetric amenability and the nonexistence of Lie and Jordan derivations*, Math. Proc. Cambridge Philos. Soc. **120** (1996) 455–473.
6. F. Lu, *Characterization of derivations and Jordan derivations on Banach algebras*, Linear Algebra Appl. **430** (2009) 2233–2239.
7. T. Palmer, *Banach algebras and the general theory of $*$-algebras*, Vol I. Cambridge University Press, Cambridge, 1994.
8. A. Zivari-Kazempour, *A characterization of* 3*-Jordan homomorphisms on Banach algebras*, Bull. Aust. Math. Soc. **93** (2016), no. 2, 301–306.
9. A. Zivari-Kazempour, *Automatic continuity of n-Jordan homomorphisms on Banach algebras*, Commun. Korean Math. Soc. **33**(1) (2018) 165–170.
10. A. Zivari-Kazempour, *Characterization of Pseudo n-Jordan homomorphisms*, Kyungpook Math. J. **61** (2021) 734–744.

Department of Mathematics, Ayatollah Borujerdi University, Borujerd, Iran.
Email address: zivari@abru.ac.ir; zivari6526@gmail.com