

# Khayyam Journal of Mathematics 

emis.de/journals/KJM kjm-math.org

# VISCOSITY LIKE IMPLICIT METHODS FOR ZEROS OF MONOTONE OPERATORS IN BANACH SPACES 

JOHN T. MENDY ${ }^{1}$ AND RAHUL SHUKLA ${ }^{2 *}$<br>Communicated by A.M. Peralta


#### Abstract

We present some implicit methods to approximate the zeros of monotone operators in the setting of Banach spaces. The methods considered herein converge strongly to the desired solutions under certain assumptions. As applications, we employ our methods to obtain solutions of convex minimization problems and Fredholm integral equations. Finally, we show the effectiveness and efficiency of the algorithm considered herein.


## 1. Introduction

Throughout this paper, $\mathcal{M}$ is a real Banach space having dual $\mathcal{M}^{*}$. Let $J$ be a normalized duality mapping from $\mathcal{M}$ into $2^{\mathcal{M}^{*}}$ as follows:

$$
J(u):=\left\{\lambda^{*} \in \mathcal{M}^{*}:\left\langle u, \lambda^{*}\right\rangle=\|u\|\left\|\lambda^{*}\right\|,\|u\|=\left\|\lambda^{*}\right\|\right\}
$$

where $\langle\cdot, \cdot\rangle$ is used as a generalized duality pairing between $\mathcal{M}$ and $\mathcal{M}^{*}$. For more properties of normalized duality mapping, see [8]. A mapping $G \subset \mathcal{M} \times \mathcal{M}^{*}$ with domain $\mathcal{D}(G)=\{u \in \mathcal{M}: G(u) \neq 0\}$ and range $\mathcal{R}(G)=\cup\{G(u) \in \mathcal{M}: u \in$ $\mathcal{D}(G)\}$ is said to be monotone if

$$
\left\langle u-v, u^{*}-v^{*}\right\rangle \geq 0
$$

for all $\left(u, u^{*}\right),\left(v, v^{*}\right) \in G$. To ensure the solution of nonlinear evolution equation, Browder [6] and Kato [11] introduced new operators, namely accretive operator,

[^0]independently. A mapping $G: \mathcal{D}(G) \subset \mathcal{M} \rightarrow \mathcal{R}(G) \subset \mathcal{M}$ is said to be accretive if for every $u, v \in \mathcal{D}(G)$, there exists $j(u-v) \in J(u-v)$ such that
$$
\langle G(u)-G(v), j(u-v)\rangle \geq 0
$$

For more details on accretive operators, see [11]. If a Banach space $\mathcal{M}$ is reduced to a Hilbert space, then an accretive operator is equivalent to a monotone operator in terms of Browder [6] and Minty [13].

It is widely known fact that a number of important problems arising in various fields can be modeled in the form of initial value problem of the form

$$
\begin{equation*}
\frac{d u}{d t}+G(u)=0, u(0)=u_{0} \tag{1.1}
\end{equation*}
$$

where $G$ is an accretive operator on the underline Banach space. The models of heat, wave, and Schrödinger equations are some examples of such evolution equations (see [7]). Browder [6] proved that (1.1) is solvable if $G$ is an accretive operator and locally Lipschitzian on the underline Banach space. Numerous mathematicians obtained the solution of (1.1) under different assumptions on the operator $G$ (see [12]).

It easily implies that if $u$ is independent of $t$ in evolution equation (1.1), then $\frac{d u}{d t}=0$ and (1.1) becomes $G(u)=0$. Thus the equilibrium points of the system described by (1.1) are corresponding to approximating zeros of accretive operators; see $[6,8]$ and references therein. Furthermore, the problem of approximating zero of monotone operator is related with the convex minimization problem; see [23] for more details. Therefore, approximating a zero of monotone (or accretive) operator is a paramount problem in the field of nonlinear analysis and optimization.

To obtain a zero of an accretive operator in the setting of Hilbert space $\mathcal{X}$, Browder [6] considered an operator $S: \mathcal{X} \rightarrow \mathcal{X}$ by $S=I-G$, where $G$ is an accretive operator and $I$ is the identity operator on a Hilbert space $\mathcal{X}$. The above defined operator $S$ is known as pseudo-contractive, and the zero of $G$ (if it exists) is equivalent to the fixed point of $S$. Thus, obtaining the solution of $0 \in G(u)$ is reduced to approximating the fixed points of pseudo-contractive mappings. Therefore, in the last 30 years or so, a number of papers have been appeared in literature dealing with some new iterative techniques for obtaining fixed points of pseudo-contractive mappings (or, equivalently, zeros of accretive mappings); see [8] and reference therein.

However, it can be easily seen that the methods of converting the inclusion problem $0 \in G(u)$ into a fixed point problem for $(I-G): \mathcal{M} \rightarrow \mathcal{M}$ is not relevant in the setting of Banach spaces, since, when $G$ is monotone, the identity mapping and a mapping from $\mathcal{M}$ into $\mathcal{M}^{*}$ does not make sense (see $[12,21]$ and others). Considering this fact, many authors studied algorithms to approximate solutions of equations $0 \in G(u)$ when $G: \mathcal{M} \rightarrow \mathcal{M}^{*}$ is of monotone type in Banach spaces and this field is flourishing in the nonlinear analysis; see [9,10, 23].

On the other hand, Moudafi [14] considered the following iterative algorithm: Let $\mathcal{X}$ be a Hilbert space and let $\mathcal{Y}$ be a closed convex subset of $\mathcal{X}$. Let $\psi: \mathcal{Y} \rightarrow \mathcal{Y}$ be a contraction and let $S: \mathcal{Y} \rightarrow \mathcal{Y}$ be a nonexpansive mapping. For all $n \in \mathbb{N}$,
$\beta_{n} \in(0,1), u_{1} \in \mathcal{Y}$, and the sequence $\left\{u_{n}\right\}$ is defined by

$$
\begin{equation*}
u_{n+1}=\beta_{n} \psi\left(u_{n}\right)+\left(1-\beta_{n}\right) S\left(u_{n}\right) . \tag{1.2}
\end{equation*}
$$

The sequence (1.2) under certain assumptions converges strongly to a fixed point of $S$, which is also a solution of the following the variational inequality:

$$
\langle(I-\psi) \bar{u}, \bar{u}-u\rangle \leq 0 \quad \text { for all } u \in F(S)
$$

where $F(S)$ is the set of fixed points of $S$. The algorithm (1.2) considered by Moudafi is known as viscosity approximation. These methods have been enormously studied in the literature in order to guarantee strong convergence of the sequence (see [15, 17, 20, 22] and references therein).

The implicit midpoint rule (IMR) is one of the powerful methods for solving ordinary differential equations; see $[3,18]$ and the references therein. Considering this fact, Xu, Alghamdi, and Shahzad [22] obtained the following implicit midpoint viscosity approximation method.
Theorem 1.1. Let $\mathcal{X}$ be a Hilbert space and let $\mathcal{Y}$ be a closed convex subset of $\mathcal{X}$. Let $S: \mathcal{Y} \rightarrow \mathcal{Y}$ be a nonexpansive mapping with $F(S) \neq \emptyset$. Let $\psi: \mathcal{Y} \rightarrow \mathcal{Y}$ be a contraction with coefficient $\rho \in[0,1)$. For given $u_{1} \in \mathcal{Y}$, the sequence $\left\{u_{n}\right\}$ is defined by

$$
u_{n+1}=\beta_{n} \psi\left(u_{n}\right)+\left(1-\beta_{n}\right) S\left(\frac{u_{n}+u_{n+1}}{2}\right), \quad n \geq 1
$$

satisfying the following assumptions:
(C1) $\lim _{n \rightarrow \infty} \beta_{n}=0$,
(C2) $\sum_{n=0}^{\infty} \beta_{n}=\infty$,
(C3) either $\sum_{n=0}^{\infty}\left|\beta_{n+1}-\beta_{n}\right|<\infty$ or $\lim _{n \rightarrow \infty} \frac{\beta_{n+1}}{\beta_{n}}=1$.
Then the sequence $\left\{u_{n}\right\}$ converges strongly to a fixed point of $S$ that is also a solution of the following the variational inequality:

$$
\langle(I-\psi) \bar{u}, \bar{u}-u\rangle \leq 0 \quad \text { for all } u \in F(S)
$$

Recently, motivated by Xu, Alghamdi, and Shahzad [22], Tang and Bao [20] considered the semi-implicit midpoint rule (SIMR) and obtained the following result.

Theorem 1.2. Let $\mathcal{M}$ be a 2-uniformly smooth and uniformly convex Banach space with dual $\mathcal{M}^{*}$. Let $G: \mathcal{M}^{*} \rightarrow \mathcal{M}$ be an L-Lipschitz continuous monotone mapping such that $G^{-1}(0) \neq \emptyset$. Let $\psi: \mathcal{M} \rightarrow \mathcal{M}$ be a contraction with coefficient $\rho \in(0,1)$ and let $I: \mathcal{M} \rightarrow \mathcal{M}$ be an identity mapping. For given $u_{1} \in \mathcal{M}$, the sequence is defined by

$$
u_{n+1}=\alpha_{n} \psi\left(u_{n}\right)+\beta_{n} u_{n}+\gamma_{n}\left(I-\omega_{n} G J\right)\left(\frac{u_{n}+u_{n+1}}{2}\right)
$$

where $J$ is the normalized duality mapping from $\mathcal{M} \rightarrow \mathcal{M}^{*}$, and the sequences $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\},\left\{\gamma_{n}\right\}$, and $\left\{\omega_{n}\right\}$ are in the interval $[0,1]$ satisfying the following assumptions:
(1) $\alpha_{n}+\beta_{n}+\gamma_{n}=1, \lim _{n \rightarrow \infty} \alpha_{n}=0$, and $\sum_{n=0}^{\infty} \alpha_{n}=\infty$,
(2) $\lim _{n \rightarrow \infty} \frac{\omega_{n}}{\alpha_{n}}=0$ and $\sum_{n=0}^{\infty} \omega_{n}<\infty$.

Assume $\mathcal{K}_{\text {min }} \cap(G J)^{-1}(0) \neq \emptyset$. Then the sequence $\left\{u_{n}\right\}$ converges strongly to an element $u^{\dagger} \in(G J)^{-1}(0)$.

Motivated by Xu, Alghamdi, and Shahzad [22], Tang and Bao [20], and others, we consider an implicit viscosity like an algorithm to approximate the zero of a monotone operator in the setting of a Banach space. Some new algorithms are suggested to approximate the solutions of convex minimization problems and Fredholm integral equations. Finally, we present the effectiveness and efficiency of the algorithm considered herein over SIMR. This way, results in [9, 20, 23] are complemented, extended, and generalized.

## 2. Preliminaries

A Banach space $\mathcal{M}$ is said to be uniformly convex if for each $\varepsilon \in(0,2]$, there exists $\delta>0$ such that $\left\|\frac{u+v}{2}\right\| \leq 1-\delta$ for all $u, v \in \mathcal{M}$ with $\|u\|=\|v\|=1$ and $\|u-v\|>\varepsilon$. The modulus of smoothness of a Banach space $\mathcal{M}$ (denoted by $\left.\rho_{\mathcal{M}}:[0, \infty) \rightarrow[0, \infty)\right)$ is defined by

$$
\rho_{\mathcal{M}}(s)=\sup \left\{\frac{\|u+s v\|+\|u-s v\|}{2}-1:\|u\|=\|v\|=1\right\}, \quad s \geq 0
$$

A Banach space $\mathcal{M}$ is said to be uniformly smooth if $\lim _{s \rightarrow 0} \frac{\rho_{\mathcal{M}}(s)}{s}=0$. A Banach space is said to be $q$-uniformly smooth if there exists a constant $c>0$ such that $\rho_{\mathcal{M}}(s) \leq c s^{q}$ for all $s>0$, where $q>1$ is a fixed real number. It is well known that every $q$-uniformly smooth Banach space is uniformly smooth. Every uniformly smooth Banach space is smooth. If a Banach space $\mathcal{M}$ is smooth, then the duality mapping $J: \mathcal{M} \rightarrow 2^{\mathcal{M}^{*}}$ is single-valued. If $p \geq 2$, then $L_{p}$ or $W^{m p}$ is 2-uniformly smooth.

The norm of $\mathcal{M}$ is said to be uniformly Gâteaux differentiable if for each $v \in$ $S_{\mathcal{M}}:=\{u \in \mathcal{M}:\|u\|=1\}$, the limit

$$
\lim _{s \rightarrow 0} \frac{\|u+s v\|-\|u\|}{s}
$$

exists (uniformly) for $u \in S_{\mathcal{M}}$. If a Banach space $\mathcal{M}$ has a uniformly Gâteaux differentiable norm, then the duality mapping is norm to weak* uniformly continuous; that is, if any sequence $\left\{u_{n}\right\}$ in $\mathcal{M}$ converges strongly to $u$, then $J\left(u_{n}\right)$ converges to $J(u)$ in weak* topology. For more details on the geometry of Banach spaces, one may refer to [1].

Let $\ell^{\infty}$ be the Banach space of bounded real sequences with the supremum norm. It is well known that there exists a bounded linear functional $\mu$ on $\ell^{\infty}$ such that the following three conditions hold:
(a) If $t_{n} \in \ell^{\infty}$ and $t_{n} \geq 0$ for every $n \in \mathbb{N}$, then $\mu\left(\left\{t_{n}\right\}\right) \geq 0$;
(b) if $t_{n}=1$ for every $n \in \mathbb{N}$, then $\mu\left(\left\{t_{n}\right\}\right)=1$;
(c) $\mu\left(\left\{t_{n}\right\}\right)=\mu\left(\left\{t_{n}+1\right\}\right)$ for all $\left\{t_{n}\right\} \in \ell^{\infty}$.

Such a functional $\mu$ is called a Banach limit, and the value of $\mu$ at $\left\{t_{n}\right\} \in \ell^{\infty}$ is denoted by $\mu_{n} t_{n}$.
Let $\left\{u_{n}\right\}$ be a bounded sequence in $\mathcal{M}$. For sufficiently large $R>0$, we have $u_{n} \in \mathcal{K}:=\overline{B_{R}\left(u^{*}\right)}$ for all $n \in \mathbb{N}$, where $u^{*} \in \mathcal{M}$. It is noted that $\mathcal{K}$ is a nonempty, bounded, closed and convex subset of $\mathcal{M}$. Let $f$ be a real valued function on $\mathcal{M}$ defined as follows:

$$
f(u)=\mu_{n}\left\|u_{n}-u\right\|^{2} \quad \text { for all } u \in \mathcal{M}
$$

Then the function $f$ is a convex and continuous. If $\mathcal{M}$ is reflexive, then there exists $v \in \mathcal{M}$ such that $f(v)=\min _{u \in \mathcal{K}} f(u)$. Let $\mathcal{K}_{\text {min }}$ be a set defined as follows:

$$
\mathcal{K}_{\text {min }}:=\left\{v \in \mathcal{K}: f(v)=\min _{u \in \mathcal{K}} f(u)\right\} .
$$

It is noted that $\mathcal{K}_{\text {min }}$ is a nonempty closed convex bounded subset of $\mathcal{M}$; see [23].
Proposition 2.1 ([16]). Let a be a real number and let $\left(a_{1}, a_{2}, \ldots\right) \in \ell^{\infty}$ such that $\mu_{n}\left(a_{n}\right) \leq a$ for all Banach limits $\mu$ and $\limsup \left(a_{n+1}-a_{n}\right) \leq 0$. Then $\limsup _{n \rightarrow \infty} a_{n} \leq a$.
Lemma 2.2 ([19]). Let $\left\{\zeta_{n}\right\} \subset \mathbb{R}^{+}$(set of nonnegative real numbers) such that for all $n \geq 0$

$$
\zeta_{n+1}=\left(1-\theta_{n}\right) \zeta_{n}+\eta_{n},
$$

where $\left\{\theta_{n}\right\}$ and $\left\{\eta_{n}\right\}$ are real sequences such that
(i) $\lim _{n \rightarrow \infty} \theta_{n}=0, \quad \sum_{n=1}^{\infty} \theta_{n}=\infty$,
(ii) $\lim _{n \rightarrow \infty} \frac{\eta_{n}}{\theta_{n}} \leq 0, \quad \sum_{n=1}^{\infty} \eta_{n}<\infty$.

Then the sequence $\left\{\zeta_{n}\right\}$ converges to 0 .
In [2], it was shown that if $\mathcal{M}$ is 2-uniformly smooth, then there exists a constant $L_{1}>0$, such that for all $u, v \in \mathcal{M}$,

$$
\|J(u)-J(v)\| \leq L_{1}\|u-v\|
$$

Definition 2.3. A mapping $G: \mathcal{D}(G) \subset \mathcal{M}^{*} \rightarrow \mathcal{M}$ is said to be $L$-Lipschitz continuous if there exists a number $L>0$, such that for all $u, v \in D(G)$,

$$
\|G(u)-G(v)\|_{\mathcal{M}} \leq L\|u-v\|_{\mathcal{M}^{*}} .
$$

We denote the set of zeros of $G$ by $G^{-1}(0):=\{u \in D(G): 0 \in G(u)\}$.
Definition 2.4. A mapping $\psi: \mathcal{M} \rightarrow \mathcal{M}$ is called a contraction with a coefficient $\rho \in[0,1)$ if, for all $u, v \in \mathcal{M}$,

$$
\|\psi(u)-\psi(v)\| \leq \rho\|u-v\| .
$$

Definition 2.5. The mapping $S: \mathcal{M} \rightarrow \mathcal{M}$ is said to be nonexpansive if, for all $u, v \in \mathcal{M}$,

$$
\|S(u)-S(v)\| \leq\|u-v\| .
$$

Lemma 2.6 ([21]). Let $\mathcal{M}$ be a real Banach space with dual $\mathcal{M}^{*}$ and let $J$ : $\mathcal{M} \rightarrow 2^{\mathcal{M}^{*}}$ be the generalized duality pairing. Then

$$
\begin{equation*}
\|u+v\|^{2} \leq\|u\|^{2}+2\langle v, j(u+v)\rangle \tag{2.1}
\end{equation*}
$$

for all $u, v \in \mathcal{M}$ and $j(u+v) \in J(u+v)$.
Definition 2.7 ([4]). Let $\mathcal{M}$ be a normed linear space. A functional $g: \mathcal{M}: \rightarrow \mathbb{R}$ is weakly lower semicontinuous if, for every sequence $\left\{u_{n}\right\} \subset \mathcal{M}$ converging weakly to $u \in \mathcal{M}$, we have

$$
g(u) \leq \lim \inf _{n \rightarrow \infty} g\left(u_{n}\right)
$$

## 3. Main Results

Now we present our main theorem.
Theorem 3.1. Let $\mathcal{M}$ be a 2-uniformly smooth and uniformly convex Banach space having dual $\mathcal{M}^{*}$. Let $G: \mathcal{M}^{*} \rightarrow \mathcal{M}$ be an L-Lipschitz continuous monotone mapping such that $G^{-1}(0) \neq 0$. Let $\psi: \mathcal{M} \rightarrow \mathcal{M}$ be a contraction mapping with coefficient $\rho \in(0,1)$ and let $I: \mathcal{M} \rightarrow \mathcal{M}$ be an identity mapping. Let $\left\{s_{n}\right\}$ be a sequence in $(s, 1]$ with $s \in(0,1)$. For a given $u_{1} \in \mathcal{M}$, define the sequence $\left\{u_{n}\right\}$ as follows:

$$
\begin{equation*}
u_{n+1}=\alpha_{n} \psi\left(u_{n}\right)+\beta_{n} u_{n}+\gamma_{n}\left(I-\omega_{n} G J\right)\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right), \tag{3.1}
\end{equation*}
$$

where $J$ is the normalized duality mapping from $\mathcal{M}$ into $\mathcal{M}^{*}$ and the sequences $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\},\left\{\gamma_{n}\right\}$, and $\left\{\omega_{n}\right\}$ are in the interval $[0,1]$ satisfying the following assumptions:

$$
\begin{aligned}
& \text { (H1) } \alpha_{n}+\beta_{n}+\gamma_{n}=1, \lim _{n \rightarrow \infty} \alpha_{n}=0 \text {, and } \sum_{n=0}^{\infty} \alpha_{n}=\infty ; \\
& \text { (H2) } \lim _{n \rightarrow \infty} \frac{\omega_{n}}{\alpha_{n}}=0 \text { and } \sum_{n=0}^{\infty} \omega_{n}<\infty .
\end{aligned}
$$

Assume that $\mathcal{K}_{\min } \cap(G J)^{-1}(0) \neq \emptyset$. Then the sequence $\left\{u_{n}\right\}$ strongly converges to an element $u^{\dagger} \in(G J)^{-1}(0)$.
Proof. First, we show that the sequence $\left\{u_{n}\right\}$ is bounded. Let $u^{\dagger} \in(G J)^{-1}(0)$ or $J\left(u^{\dagger}\right) \in G^{-1}(0)$. Let $a:=\min \left\{1-\gamma_{n}+\gamma_{n} s\right\}>0$. Since $\lim _{n \rightarrow \infty} \alpha_{n}=0$ and $\lim _{n \rightarrow \infty} \frac{\omega_{n}}{\alpha_{n}}=0$, there exists $N_{0} \in \mathbb{N}$ such that $\alpha_{n} \leq \frac{a}{4}$ and $\frac{\omega_{n}}{\alpha_{n}} \leq \frac{1}{4 L L_{1}}$ for all $n \geq N_{0}$. Let $r>0$ be sufficient large such that $u_{N_{0}} \in B_{r}\left(u^{\dagger}\right)$ and $f\left(u_{N_{0}}\right) \in B_{\frac{r}{5}}\left(u^{\dagger}\right)$. Now, we show that the sequence $\left\{u_{n}\right\} \subseteq B=\overline{B_{r}\left(u^{\dagger}\right)}$ for all integers $n \geq N_{0}$. We show this by induction. We suppose that for any $n>N_{0}, u_{n} \in B$ and show that $u_{n+1} \in B$. For this, we assume that $\left\|u_{n+1}-u^{\dagger}\right\|>r$. From (3.1), we have
$u_{n+1}-u_{n}=\alpha_{n}\left(\psi\left(u_{n}\right)-u_{n}\right)+\gamma_{n}\left(1-s_{n}\right)\left(u_{n+1}-u_{n}\right)-\gamma_{n} \omega_{n} G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right)$.
It follows that

$$
\left(1-\gamma_{n}\left(1-s_{n}\right)\right)\left(u_{n+1}-u_{n}\right)=\alpha_{n}\left(\psi\left(u_{n}\right)-u_{n}\right)-\gamma_{n} \omega_{n} G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right)
$$

and

$$
u_{n+1}-u_{n}=\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left(\psi\left(u_{n}\right)-u_{n}\right)
$$

$$
\begin{equation*}
-\frac{\gamma_{n} \omega_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)} G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right) . \tag{3.2}
\end{equation*}
$$

From (3.2) and Lemma 2.6, we get

$$
\begin{aligned}
\left\|u_{n+1}-u^{\dagger}\right\|^{2}= & \left\|u_{n+1}-u_{n}+u_{n}-u^{\dagger}\right\|^{2} \\
\leq & \left\|u_{n}-u^{\dagger}\right\|^{2}+2\left\langle u_{n+1}-u_{n}, j\left(u_{n+1}-u^{\dagger}\right)\right\rangle \\
= & \left\|u_{n}-u^{\dagger}\right\|^{2}+2\left\langle\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left(\psi\left(u_{n}\right)-u_{n}\right)\right. \\
& \left.-\frac{\gamma_{n} \omega_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)} G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right), j\left(u_{n+1}-u^{\dagger}\right)\right\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|u_{n+1}-u^{\dagger}\right\|^{2}= & \left\|u_{n}-u^{\dagger}\right\|^{2}+2\left\langle\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left(\psi\left(u_{n}\right)-u_{n}\right)\right. \\
& -\frac{\gamma_{n} \omega_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)} G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right) \\
& +\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left(u_{n+1}-u^{\dagger}\right) \\
& \left.-\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left(u_{n+1}-u^{\dagger}\right), j\left(u_{n+1}-u^{\dagger}\right)\right\rangle \\
= & \left\|u_{n}-u^{\dagger}\right\|^{2}-\frac{2 \alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left\|u_{n+1}-u^{\dagger}\right\|^{2} \\
& +2\left\langle\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left(\psi\left(u_{n}\right)-u^{\dagger}\right)\right. \\
& -\frac{\gamma_{n} \omega_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)} G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right) \\
& \left.+\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left(u_{n+1}-u_{n}\right), j\left(u_{n+1}-u^{\dagger}\right)\right\rangle
\end{aligned}
$$

Using (3.2) in the above inequality, we have

$$
\begin{aligned}
\left\|u_{n+1}-u^{\dagger}\right\|^{2} \leq & \left\|u_{n}-u^{\dagger}\right\|^{2}-\frac{2 \alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left\|u_{n+1}-u^{\dagger}\right\|^{2} \\
& +2\left\langle\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left(\psi\left(u_{n}\right)-u^{\dagger}\right)\right. \\
& -\frac{\gamma_{n} \omega_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)} G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right) \\
& +\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left\{\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left(\psi\left(u_{n}\right)-u_{n}\right)\right. \\
& \left.-\frac{\gamma_{n} \omega_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)} G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right)\right\} \\
& \left., j\left(u_{n+1}-u^{\dagger}\right)\right\rangle
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|u_{n+1}-u^{\dagger}\right\|^{2} \leq & \left\|u_{n}-u^{\dagger}\right\|^{2}-\frac{2 \alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left\|u_{n+1}-u^{\dagger}\right\|^{2}+ \\
& 2\left\langle\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left(\psi\left(u_{n}\right)-u^{\dagger}\right)\right. \\
& -\frac{\gamma_{n} \omega_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)} G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right) \\
& +\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left\{\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left(\psi\left(u_{n}\right)-u^{\dagger}\right)\right. \\
& -\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left(u_{n}-u^{\dagger}\right) \\
& \left.-\frac{\gamma_{n} \omega_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)} G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right)\right\} \\
& \left., j\left(u_{n+1}-u^{\dagger}\right)\right\rangle .
\end{aligned}
$$

Thus

$$
\begin{aligned}
\| u_{n+1}- & u^{\dagger} \|^{2} \\
\leq & \left\|u_{n}-u^{\dagger}\right\|^{2}-\frac{2 \alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left\|u_{n+1}-u^{\dagger}\right\|^{2}+2\left[\left(\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\right.\right. \\
& \left.+\frac{\alpha_{n}^{2}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)^{2}}\right)\left\|\psi\left(u_{n}\right)-u^{\dagger}\right\|+\frac{\alpha_{n}^{2}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)^{2}}\left\|u_{n}-u^{\dagger}\right\| \\
& +\left(\frac{\gamma_{n} \omega_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}+\frac{\gamma_{n} \omega_{n} \alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)^{2}}\right) \\
& \left.\left\|G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right)-G J\left(u^{\dagger}\right)\right\|\right]\left\|u_{n+1}-u^{\dagger}\right\| .
\end{aligned}
$$

Since $\left\|u_{n+1}-u^{\dagger}\right\|-\left\|u_{n}-u^{\dagger}\right\|>0$ and for the fact that $G$ and $J$ are Lipschitz, we have the following estimate:

$$
\begin{aligned}
& \frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left\|u_{n+1}-u^{\dagger}\right\| \\
& \leq\left(\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}+\frac{\alpha_{n}^{2}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)^{2}}\right)\left\|\psi\left(u_{n}\right)-u^{\dagger}\right\| \\
&+\frac{\alpha_{n}^{2}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)^{2}}\left\|u_{n}-u^{\dagger}\right\|+\left(\frac{\gamma_{n} \omega_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\right. \\
&\left.+\frac{\gamma_{n} \omega_{n} \alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)^{2}}\right)\left\|G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right)-G J\left(u^{\dagger}\right)\right\|
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\left\|u_{n+1}-u^{\dagger}\right\| \leq & \left(1+\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\right)\left\|\psi\left(u_{n}\right)-u^{\dagger}\right\| \\
& +\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left\|u_{n}-u^{\dagger}\right\| \\
& +\left(\frac{\gamma_{n} \omega_{n}}{\alpha_{n}}+\frac{\gamma_{n} \omega_{n} \alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right) \alpha_{n}}\right) \\
& L L_{1}\left\|s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}-u^{\dagger}\right\|
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|u_{n+1}-u^{\dagger}\right\| \leq & \left(1+\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\right)\left\|\psi\left(u_{n}\right)-u^{\dagger}\right\| \\
& +\frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left\|u_{n}-u^{\dagger}\right\| \\
& +\frac{\omega_{n}}{\alpha_{n}}\left(\gamma_{n}+\frac{\gamma_{n} \alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\right) \\
& L L_{1}\left(s_{n}\left\|u_{n}-u^{\dagger}\right\|+\left(1-s_{n}\right)\left\|u_{n+1}-u^{\dagger}\right\|\right)
\end{aligned}
$$

From the assumptions on the sequences $\left\{\alpha_{n}\right\},\left\{\omega_{n}\right\},\left\{u_{n}\right\}$, and $\left\{\psi\left(u_{n}\right)\right\}$, we have

$$
\left\|u_{n+1}-u^{\dagger}\right\| \leq \frac{r}{4}+\frac{r}{4}+\frac{1}{4 L L_{1}} 2 L L_{1}\left(s_{n} r+\left(1-s_{n}\right)\left\|u_{n+1}-u^{\dagger}\right\|\right)
$$

It follows that

$$
\left\|u_{n+1}-u^{\dagger}\right\| \leq r
$$

a contradiction. Thus, the sequence $\left\{u_{n}\right\}$ is in $B$ from all integers $n \geq N_{0}$ and $\left\{u_{n}\right\}$ is bounded. Therefore, the sequences $\left\{\psi\left(u_{n}\right)\right\}$ and $\left\{G J\left(u_{n}\right)\right\}$ are bounded. Furthermore, we prove that $\lim _{n \rightarrow \infty}\left\|u_{n+1}-u_{n}\right\|=0$. From (3.2), we have

$$
\begin{aligned}
\left\|u_{n+1}-u_{n}\right\|= & \frac{\alpha_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left\|\psi\left(u_{n}\right)-u_{n}\right\| \\
& +\frac{\gamma_{n} \omega_{n}}{\left(1-\gamma_{n}\left(1-s_{n}\right)\right)}\left\|G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right)\right\| .
\end{aligned}
$$

Since $\alpha_{n} \rightarrow 0$ as $n \rightarrow \infty$ and $\omega_{n}=o\left(\alpha_{n}\right)$, it follows that $\lim _{n \rightarrow \infty}\left\|u_{n+1}-u_{n}\right\|=0$. Now, we prove that $\limsup _{n \rightarrow \infty}\left\langle\psi\left(u_{n}\right)-u^{\dagger}, j\left(u_{n+1}-u^{\dagger}\right)\right\rangle \leq 0$, where $u^{\dagger} \in \mathcal{K}_{\text {min }} \cap$ $(G J)^{-1}(0)$. Since the sequences $\left\{u_{n}\right\}$ and $\left\{\psi\left(u_{n}\right)\right\}$ are bounded, there exists $\nu>0$ (sufficiently large) such that $\psi\left(u_{n}\right), u_{n} \in B_{1}:=B_{\nu}\left(u^{\dagger}\right)$ (open ball wit radius $\nu$ and center $u^{\dagger}$ ) for all $n \in N$. Furthermore, the set $B_{1}$ is a bounded closed and convex nonempty subset of $\mathcal{M}$. By the convexity of $B_{1}$, we have $(1-t) u^{\dagger}+t \psi\left(u_{n}\right) \in B_{1}$, where $t \in(0,1)$. Then, it follows from the definition of $f$ that $f\left(u^{\dagger}\right) \leq f((1-$ $\left.t) u^{\dagger}+t \psi\left(u_{n}\right)\right)$. Using Lemma 2.6, we have
$\left\|u_{n}-u^{\dagger}-t\left(\psi\left(u_{n}\right)-u^{\dagger}\right)\right\|^{2} \leq\left\|u_{n}-u^{\dagger}\right\|^{2}-2 t\left\langle\psi\left(u_{n}\right)-u^{\dagger}, j\left(u_{n}-u^{\dagger}-t\left(\psi\left(u_{n}\right)-u^{\dagger}\right)\right)\right\rangle$.

Thus taking Banach limit over $n \in \mathbb{N}$ implies

$$
\begin{aligned}
\mu_{n}\left\|u_{n}-u^{\dagger}-t\left(\psi\left(u_{n}\right)-u^{\dagger}\right)\right\|^{2} \leq & \mu_{n}\left\|u_{n}-u^{\dagger}\right\|^{2} \\
& -2 t \mu_{n}\left\langle\psi\left(u_{n}\right)-u^{\dagger}, j\left(u_{n}-u^{\dagger}-t\left(\psi\left(u_{n}\right)-u^{\dagger}\right)\right)\right\rangle,
\end{aligned}
$$

which means that

$$
\begin{aligned}
2 t \mu_{n} & \left\langle\psi\left(u_{n}\right)-u^{\dagger}, j\left(u_{n}-u^{\dagger}-t\left(\psi\left(u_{n}\right)-u^{\dagger}\right)\right)\right\rangle \\
& \leq \mu_{n}\left\|u_{n}-u^{\dagger}\right\|^{2}-\mu_{n}\left\|u_{n}-u^{\dagger}-t\left(\psi\left(u_{n}\right)-u^{\dagger}\right)\right\|^{2} \\
& =f\left(u^{\dagger}\right)-f\left(u^{\dagger}+t\left(\psi\left(u_{n}\right)-u^{\dagger}\right)\right) \leq 0 .
\end{aligned}
$$

Thus,

$$
\mu_{n}\left\langle\psi\left(u_{n}\right)-u^{\dagger}, j\left(u_{n}-u^{\dagger}-t\left(\psi\left(u_{n}\right)-u^{\dagger}\right)\right)\right\rangle \leq 0 .
$$

By using the weakly lower semi-continuity of the norm on $\mathcal{M}$, we have
$\left.\left\langle\psi\left(u_{n}\right)-u^{\dagger}, j\left(u_{n}-u^{\dagger}\right)\right\rangle-\left\langle\psi\left(u_{n}\right)-u^{\dagger}, j\left(u_{n}-u^{\dagger}\right)-t\left(\psi\left(u_{n}\right)-u^{\dagger}\right)\right)\right\rangle \rightarrow 0$ as $t \rightarrow 0$.
Therefore, for all $\varepsilon>0$, there exists $\delta>0$ such that for all $t \in(0, \delta)$ and $n \in \mathbb{N}$,

$$
\left.\left\langle\psi\left(u_{n}\right)-u^{\dagger}, j\left(u_{n}-u^{\dagger}\right)\right\rangle<\left\langle\psi\left(u_{n}\right)-u^{\dagger}, j\left(u_{n}-u^{\dagger}\right)-t\left(\psi\left(u_{n}\right)-u^{\dagger}\right)\right)\right\rangle+\varepsilon .
$$

Hence

$$
\left.\mu_{n}\left\langle\psi\left(u_{n}\right)-u^{\dagger}, j\left(u_{n}-u^{\dagger}\right)\right\rangle<\mu_{n}\left\langle\psi\left(u_{n}\right)-u^{\dagger}, j\left(u_{n}-u^{\dagger}\right)-t\left(\psi\left(u_{n}\right)-u^{\dagger}\right)\right)\right\rangle+\varepsilon .
$$

Since $\varepsilon$ is arbitrary small, we get

$$
\mu_{n}\left\langle\psi\left(u_{n}\right)-u^{\dagger}, j\left(u_{n}-u^{\dagger}\right)\right\rangle \leq 0 .
$$

Since the norm of $\mathcal{M}$ is uniformly Gateaux differentiable, $J$ is uniformly norm to weak* continuous on each bounded subset of $\mathcal{M}$. Then

$$
\lim _{n \rightarrow \infty}\left(\left\langle\psi\left(u_{n}\right)-u^{\dagger}, j\left(u_{n+1}-u^{\dagger}\right)\right\rangle-\left\langle\psi\left(u_{n}\right)-u^{\dagger}, j\left(u_{n}-u^{\dagger}\right)\right\rangle\right)=0
$$

Therefore, the sequence $\left\{\left\langle\psi\left(u_{n}\right)-u^{\dagger}, j\left(u_{n}-u^{\dagger}\right)\right\rangle\right\}$ satisfies all assumptions of Proposition 2.1, which implies that

$$
\limsup _{n \rightarrow \infty}\left\langle\psi\left(u_{n}\right)-u^{\dagger}, j\left(u_{n+1}-u^{\dagger}\right)\right\rangle \leq 0
$$

Finally, we show that $\left\|u_{n}-u^{\dagger}\right\| \rightarrow 0$ as $n \rightarrow \infty$, as follows:

$$
\begin{aligned}
\left\|u_{n+1}-u^{\dagger}\right\|^{2}= & \left\|u_{n+1}-u_{n}+u_{n}-u^{\dagger}\right\|^{2} \\
= & \| u_{n}-u^{\dagger}+\frac{\alpha_{n}}{1-\gamma_{n}\left(1-s_{n}\right)}\left(\psi\left(u_{n}\right)-u_{n}\right) \\
& -\frac{\gamma_{n} \omega_{n}}{1-\gamma_{n}\left(1-s_{n}\right)} G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right) \|^{2} \\
= & \|\left(1-\frac{\alpha_{n}}{1-\gamma_{n}\left(1-s_{n}\right)}\right)\left(u_{n}-u^{\dagger}\right)+\frac{\alpha_{n}}{1-\gamma_{n}\left(1-s_{n}\right)}\left(\psi\left(u_{n}\right)-u^{\dagger}\right) \\
& -\frac{\gamma_{n} \omega_{n}}{1-\gamma_{n}\left(1-s_{n}\right)} G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right) \|^{2}
\end{aligned}
$$

and

$$
\begin{aligned}
\| u_{n+1} & -u^{\dagger} \|^{2} \\
\leq & \left(1-\frac{\alpha_{n}}{1-\gamma_{n}\left(1-s_{n}\right)}\right)^{2}\left\|u_{n}-u^{\dagger}\right\|^{2}+2\left\langle\frac{\alpha_{n}}{1-\gamma_{n}\left(1-s_{n}\right)}\left(\psi\left(u_{n}\right)-u^{\dagger}\right)\right. \\
& \left.-\frac{\gamma_{n} \omega_{n}}{1-\gamma_{n}\left(1-s_{n}\right)} G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right), j\left(u_{n+1}-u^{\dagger}\right)\right\rangle \\
= & \left(1-\frac{\alpha_{n}}{1-\gamma_{n}\left(1-s_{n}\right)}\right)\left\|u_{n}-u^{\dagger}\right\|^{2}+2\left\langle\frac{\alpha_{n}}{1-\gamma_{n}\left(1-s_{n}\right)}\left(\psi\left(u_{n}\right)-u^{\dagger}\right)\right. \\
& \left.-\frac{\gamma_{n} \omega_{n}}{1-\gamma_{n}\left(1-s_{n}\right)} G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right), j\left(u_{n+1}-u^{\dagger}\right)\right\rangle .
\end{aligned}
$$

Take $Q:=\sup \left\{\left\|u_{n}-u^{\dagger}\right\|\right\}$, because $\left\{u_{n}\right\}$ is a bounded sequence. Since $G$ and $J$ are Lipschitz and continuous, we have

$$
\begin{aligned}
\left\|u_{n+1}-u^{\dagger}\right\|^{2} \leq & \left(1-\frac{\alpha_{n}}{1-\gamma_{n}\left(1-s_{n}\right)}\right)\left\|u_{n}-u^{\dagger}\right\|^{2} \\
& +\frac{2 \alpha_{n}}{1-\gamma_{n}\left(1-s_{n}\right)}\left\langle\left(\psi\left(u_{n}\right)-u^{\dagger}\right), j\left(u_{n+1}-u^{\dagger}\right)\right\rangle \\
& +\frac{2 \gamma_{n} \omega}{1-\gamma_{n}\left(1-s_{n}\right)}\left\|G J\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right)-G J\left(u^{\dagger}\right)\right\| \\
& \left\|u_{n+1}-u^{\dagger}\right\| \\
\leq & \left(1-\frac{\alpha_{n}}{1-\gamma_{n}\left(1-s_{n}\right)}\right)\left\|u_{n}-u^{\dagger}\right\|^{2}+\eta_{n}
\end{aligned}
$$

where $\eta_{n}=\frac{2 \alpha_{n}}{1-\gamma_{n}\left(1-s_{n}\right)}\left\langle\left(\psi\left(u_{n}\right)-u^{\dagger}\right), j\left(u_{n+1}-u^{\dagger}\right)\right\rangle+\frac{2 \gamma_{n} \omega}{1-\gamma_{n}\left(1-s_{n}\right)} L L_{1} Q^{2}$. Letting $\theta_{n}=\frac{\alpha_{n}}{1-\gamma_{n}\left(1-s_{n}\right)}$, then from Lemma 2.2, we obtain

$$
\left\|u_{n}-u^{\dagger}\right\|=0 \text { as } n \rightarrow \infty .
$$

Thus, the sequence $\left\{u_{n}\right\}$ converges strongly to the solution $u^{\dagger}$, and this completes the proof.

Theorem 3.2. Let $\mathcal{M}, G$, and $J$ be defined as in Theorem 3.1. Let $\left\{s_{n}\right\}$ be a sequence in $(s, 1]$ with $s \in(0,1)$. Let $u \in \mathcal{M}$, and for a given $u_{1} \in \mathcal{M}$, define the sequence $\left\{u_{n}\right\}$ by

$$
u_{n+1}=\alpha_{n} u+\beta_{n} u_{n}+\gamma_{n}\left(I-\omega_{n} G J\right)\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right),
$$

where the sequences $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\},\left\{\gamma_{n}\right\}$, and $\left\{\omega_{n}\right\}$ are in the interval $[0,1]$ satisfying assumptions (H1) and (H2). Assume that $\mathcal{K}_{\min } \cap(G J)^{-1}(0) \neq \emptyset$. Then the sequence $\left\{u_{n}\right\}$ strongly converges to an element $u^{\dagger} \in(G J)^{-1}(0)$.

Corollary 3.3. Let $\mathcal{M}, G, J$, and $\psi$ be defined as in Theorem 3.1. For a given $u_{1} \in \mathcal{M}$, define the sequence $\left\{u_{n}\right\}$ by

$$
u_{n+1}=\alpha_{n} \psi\left(u_{n}\right)+\beta_{n} u_{n}+\gamma_{n}\left(I-\omega_{n} G J\right)\left(u_{n+1}\right),
$$

where the sequences $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\},\left\{\gamma_{n}\right\}$, and $\left\{\omega_{n}\right\}$ are in the interval $[0,1]$ satisfying assumptions (H1) and (H2). Assume that $\mathcal{K}_{\min } \cap(G J)^{-1}(0) \neq \emptyset$. Then the sequence $\left\{u_{n}\right\}$ strongly converges to an element $u^{\dagger} \in(G J)^{-1}(0)$.

Corollary 3.4 ([20]). Let $\mathcal{M}, G, J$, and $\psi$ be defined as in Theorem 3.1. For given $u_{1} \in \mathcal{M}$, define the sequence $\left\{u_{n}\right\}$ by

$$
u_{n+1}=\alpha_{n} \psi\left(u_{n}\right)+\beta_{n} u_{n}+\gamma_{n}\left(I-\omega_{n} G J\right)\left(\frac{u_{n}+u_{n+1}}{2}\right)
$$

where the sequences $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\},\left\{\gamma_{n}\right\}$, and $\left\{\omega_{n}\right\}$ are in the interval $[0,1]$ satisfying assumptions (H1) and (H2). Assume that $\mathcal{K}_{\min } \cap(G J)^{-1}(0) \neq \emptyset$. Then the sequence $\left\{u_{n}\right\}$ strongly converges to an element $u^{\dagger} \in(G J)^{-1}(0)$.

Let $\mathcal{K}$ be a nonempty subset of a smooth, uniformly convex Banach space $\mathcal{M}$ with dual $\mathcal{M}^{*}$. Let $\mathcal{K}^{*}$ be the dual space of $\mathcal{K}$. A mapping $S: \mathcal{K}^{*} \rightarrow \mathcal{M}$ is said to be semi-pseudo if

$$
G J(u):=\left(J^{-1}-S\right) J(u)
$$

for all $J(u) \in \mathcal{K}^{*}$ is monotone mapping, where $J$ is the normalized duality mapping from $\mathcal{M}$ into $\mathcal{M}^{*}$. We denote the set of all semi-fixed points of $S$ by $F_{s}(S):=\left\{J(u) \in \mathcal{K}^{*}: S J(u)=u\right\}$. It is noted that a zero of a monotone mapping $G$ is a semi-fixed point of a semi-pseudo mapping $S$. If $\mathcal{M}$ is a Hilbert space, the definition of semi-pseudo and semi-fixed point of $S$ coincides with pseudo-contraction and fixed point of pseudo-contraction $S$, respectively; see [23].

Corollary 3.5. Let $\mathcal{K}$ be a nonempty closed and convex subset of a 2-uniformly smooth and uniformly convex Banach space $\mathcal{M}$ having dual $\mathcal{M}^{*}$. Let $\mathcal{K}^{*}$ be the dual space of $\mathcal{K}$. Let $S: \mathcal{K}^{*} \rightarrow \mathcal{K}$ be an L-Lipschitz continuous semi-pseudo mapping with $F_{s}(S) \neq \emptyset$ and let $G:=\left(J^{-1}-S\right)$ be a maximal monotone mapping on $\mathcal{K}^{*}$. Let $\psi: \mathcal{M} \rightarrow \mathcal{M}$ be a contraction mapping with coefficient $\rho \in(0,1)$. Let $\left\{s_{n}\right\}$ be a sequence in $(s, 1]$ with $s \in(0,1)$. For a given $u_{1} \in \mathcal{M}$, define the sequence $\left\{u_{n}\right\}$ by

$$
u_{n+1}=\alpha_{n} \psi\left(u_{n}\right)+\beta_{n} u_{n}+\gamma_{n}\left(\left(1-\omega_{n}\right) I+\omega_{n} G J\right)\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right)
$$

where $J$ is the normalized duality mapping from $\mathcal{M}$ into $\mathcal{M}^{*}$ and the sequences $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\},\left\{\gamma_{n}\right\}$, and $\left\{\omega_{n}\right\}$ are in the interval $[0,1]$ satisfying assumptions (H1) and (H2). Assume that $\mathcal{K}_{\text {min }} \cap F_{s}(S) \neq \emptyset$. Then the sequence $\left\{u_{n}\right\}$ strongly converges to $u^{\dagger}$, where $J\left(u^{\dagger}\right) \in F_{s}(S)$.

## 4. Convex minimization problem

Now, we present a convex minimization problem for a convex function $\nabla$ : $\mathcal{M} \rightarrow \mathbb{R}$.
The following results are well known.
Remark 4.1. Let $\nabla: \mathcal{M} \rightarrow \mathbb{R}$ be a differentiable convex function and let $u^{\dagger} \in \mathcal{M}$. Then the point $u^{\dagger}$ is a minimizer of $\nabla$ on $\mathcal{M}$ if and only if $d \nabla\left(u^{\dagger}\right)=0$.

Definition 4.2. A function $\nabla: \mathcal{M} \rightarrow \mathbb{R}$ is said to be strongly convex if there exists $\beta>0$ such that the following condition holds:

$$
\nabla(\mu u+(1-\mu) v) \leq \mu \nabla u+(1-\mu) \nabla v-\beta\|u-v\|^{2}
$$

for every $u, v \in \mathcal{M}$ with $u \neq v$ and $\mu \in(0,1)$.
Lemma 4.3. Let $\mathcal{M}$ be normed linear space and let $\nabla: \mathcal{M} \rightarrow \mathbb{R}$ be a convex differentiable function. Suppose that $\nabla$ is strongly convex. Then the differential map $d \nabla: \mathcal{M} \rightarrow \mathcal{M}^{*}$ is strongly monotone, that is, there exists $k>0$ such that

$$
\langle d \nabla u-d \nabla v, u-v\rangle \geq k\|u-v\|^{2} \forall u, v \in \mathcal{M} .
$$

Now we present the following result.
Theorem 4.4. Let $\mathcal{M}$ and $\psi$ be defined as in Theorem 3.1. Let $d \nabla: \mathcal{M}^{*} \rightarrow \mathcal{M}$ be an L-Lipschitz continuous monotone mapping such that $d \nabla^{-1}(0) \neq \emptyset$. Let $\left\{s_{n}\right\}$ be a sequence in $(s, 1]$ with $s \in(0,1)$. For given $u_{1} \in \mathcal{M}$, define the sequence $\left\{u_{n}\right\}$ by

$$
u_{n+1}=\alpha_{n} \psi\left(u_{n}\right)+\beta_{n} u_{n}+\gamma_{n}\left(I-\omega_{n} d \nabla J\right)\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right)
$$

where $J$ is the normalized duality mapping from $\mathcal{M}$ into $\mathcal{M}^{*}$ and the sequences $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\}$ and $\left\{\omega_{n}\right\}$ are in the interval $[0,1]$ satisfying assumptions (H1) and (H2). Assume that $\mathcal{K}_{\min } \cap(d \nabla J)^{-1}(0) \neq \emptyset$. Then the sequence $\left\{u_{n}\right\}$ converges strongly to a unique minimizer $u^{\dagger}$ of $\nabla$ in $\mathcal{M}$.

Proof. It follows from Remark 4.1 that $\nabla$ has a unique minimizer $u^{\dagger}$ and is obtained by $d \nabla\left(u^{\dagger}\right)=0$. From Lemma 4.3 and using the fact that the differential mapping $d \nabla: \mathcal{M} \rightarrow \mathcal{M}^{*}$ is Lipschitz, considering the result of Theorem 3.1, we can complete the proof.

## 5. Fredholm integral equation

Let $\mathcal{Y}=L^{2}[0,1]$ be the space of square integrable functions $u:[0,1] \rightarrow \mathbb{R}$ endowed with inner product $\langle u, v\rangle_{2}=\int_{0}^{1} u(\kappa) v(\kappa) d \kappa$. Now we discuss the solution of following Fredholm integral equation:

$$
\begin{equation*}
u(\kappa)=\phi(\kappa)+\lambda \int_{0}^{1} \tau(\kappa, \nu) \xi(\kappa, \nu, u(\nu)) d \nu, \quad \kappa, \nu \in[0,1]=V \tag{5.1}
\end{equation*}
$$

In order to present solution of above equation, we take the following postulates:
(A1) The functions $\xi: V \times V \times \mathbb{R} \rightarrow \mathbb{R}$ and $\phi: V \rightarrow \mathbb{R}$ are continuous.
(A2) $\xi$ is Lipschitz continuous, that is, for all $u, v \in \mathcal{Y}$,

$$
|\xi(\kappa, \nu, u)-\xi(\kappa, \nu, v)| \leq L|u(\kappa)-v(\kappa)|, \quad \kappa \in V
$$

(A3) $\tau: V \times V \rightarrow \mathbb{R}$ is continuous for all $(\kappa, \nu) \in V \times V,|\tau(\kappa, \nu)| \leq c$, where $c>0$.
(A4) $\lambda c L \leq 1$ and $\lambda>0$.
Now, we consider the mapping $S: \mathcal{Y} \rightarrow \mathcal{Y}$ defined as

$$
\begin{equation*}
(S u)(\kappa)=\phi(\kappa)+\lambda \int_{0}^{1} \tau(\kappa, \nu) \xi(\kappa, \nu, u(\nu)) d \nu, \quad \kappa \in[0,1]=V \tag{5.2}
\end{equation*}
$$

Now, we observe that $S$ is a nonexpansive mapping. For this, for every $u, v \in \mathcal{Y}$, we have

$$
\begin{aligned}
|S u(\kappa)-S v(\kappa)|^{2}= & \mid\left(\phi(\kappa)+\lambda \int_{0}^{1} \tau(\kappa, \nu) \xi(\kappa, \nu, u(\nu)) d \nu\right) \\
& -\left.\left(\phi(\kappa)+\lambda \int_{0}^{1} \tau(\kappa, \nu) \xi(\kappa, \nu, v(\nu)) d \nu\right)\right|^{2} \\
= & \lambda^{2}\left|\int_{0}^{1} \tau(\kappa, \nu)(\xi(\kappa, \nu, u(\nu))-\xi(\kappa, \nu, v(\nu))) d \nu\right|^{2} \\
\leq & \int_{0}^{1}|\tau(\kappa, \nu) \xi(\kappa, \nu, u(\nu))-\xi(\kappa, \nu, v(\nu))|^{2} d \nu \\
\leq & \lambda^{2} \int_{0}^{1}|\tau(\kappa, \nu)|^{2}|\xi(\kappa, \nu, u(\nu))-\xi(\kappa, \nu, v(\nu))|^{2} d \nu \\
\leq & \lambda^{2} c^{2} L^{2} \int_{0}^{1}|u(\nu)-v(\nu)|^{2} d \nu
\end{aligned}
$$

This implies that

$$
\|S(u)-S(v)\| \leq \lambda c L\|u-v\| \leq\|u-v\|
$$

and $S$ is a nonexpansive mapping. Define

$$
\mathcal{B}=\{u \in \mathcal{Y}:\|u\| \leq r\}
$$

where $r$ is sufficiently large. Then $\mathcal{B}$ is the closed ball of $\mathcal{Y}$ of radius $r$ with center at origin. It can be easily seen that $S(\mathcal{B}) \subseteq \mathcal{B}$. From [5, Theorem], the operator $S$ has a fixed point in $\mathcal{B}$, and this fixed point of operator is a solution of nonlinear integral equation (5.1).

Theorem 5.1. Let $\mathcal{Y}=L^{2}[0,1]$ be a Hilbert space defined above and let $S: \mathcal{Y} \rightarrow \mathcal{Y}$ be an operator defined in (5.2). Let $\psi: C \rightarrow C$ be a contraction with coefficient $\rho \in(0,1)$. Let $\left\{s_{n}\right\}$ be a sequence in $(s, 1]$ with $s \in(0,1)$. For a given $u_{1} \in \mathcal{M}$, define the sequence $\left\{u_{n}\right\}$ by

$$
u_{n+1}=\alpha_{n} \psi\left(u_{n}\right)+\beta_{n} u_{n}+\gamma_{n}\left(I-\omega_{n}(I-S)\right)\left(s_{n} u_{n}+\left(1-s_{n}\right) u_{n+1}\right)
$$

where $I$ is the identity mapping on $\mathcal{Y}$ and the sequences $\left\{\alpha_{n}\right\},\left\{\beta_{n}\right\},\left\{\gamma_{n}\right\}$, and $\left\{\omega_{n}\right\}$ are in the interval $[0,1]$ satisfying assumptions (H1) and (H2). Assume that $\mathcal{K}_{\text {min }} \cap(1-S)^{-1}(0) \neq \emptyset$. Then the sequence $\left\{u_{n}\right\}$ strongly converges to the solution of nonlinear integral equation (5.1).

Proof. Let $G=I-S$. Then the fixed point of the operator $S$ is a zero of the operator $G$. Now we show that operator $G$ is monotone. For this, for all $u, v \in \mathcal{Y}$ and by nonexpansiveness of $S$, we have

$$
\begin{aligned}
\langle G(u)-G(v), u-v\rangle & =\langle(I-S)(u)-(I-S)(v), u-v\rangle \\
& =\langle u-S(u)-v+S(v), u-v\rangle \\
& =\|u-v\|^{2}-\langle S(u)-S(v), u-v\rangle \\
& \geq\|u-v\|^{2}-\|S(u)-S(v)\|\|u-v\|
\end{aligned}
$$

$$
\geq\|u-v\|^{2}-\|u-v\|^{2}=0
$$

Again it can be seen that $G$ is a Lipschitz operator, that is, for all $u, v \in \mathcal{Y}$, by the nonexpansiveness of $S$, we have

$$
\begin{aligned}
\|G(u)-G(v)\| & =\|(I-S)(u)-(I-S)(v)\| \\
& =\|u-v-(S(u)-S(v))\| \leq\|u-v\|+\|(S(u)-S(v))\| \\
& \leq\|u-v\|+\|u-v\|=2\|u-v\|
\end{aligned}
$$

Therefore, in view of Theorem 3.1, the required conclusion follows.
Example 5.2. Let us consider the following nonlinear integral equation:

$$
\begin{equation*}
u(\kappa)=\left[\sin \left(\frac{\pi \kappa}{2}\right)-\frac{4}{3 \pi}\left(1+\frac{1}{\pi}\right) \kappa\right]+\int_{0}^{1} \frac{\kappa(2+\nu) u(\nu)}{3} d \nu, \quad \kappa \in[0,1] . \tag{5.3}
\end{equation*}
$$

It easily follows that (5.3) is a particular case of (5.1) with

$$
\phi(\kappa)=\sin \left(\frac{\pi \kappa}{2}\right)-\frac{4}{3 \pi}\left(1+\frac{1}{\pi}\right) \kappa \text { and } \psi(\kappa, \nu, u)=\frac{\kappa(2+\nu) u(\nu)}{3} .
$$

For any $u, v \in \mathcal{Y}$ and $\kappa, \nu \in[0,1]$, we obtain

$$
\begin{aligned}
|\psi(\kappa, \nu, u)-\psi(\kappa, \nu, v)| & =\left|\frac{\kappa(2+\nu) u}{3}-\frac{\kappa(2+\nu) v}{3}\right| \\
& \leq \frac{\kappa(2+\nu)}{3}|u-v| \leq|u-v|
\end{aligned}
$$

It is quite natural that $\phi: J \rightarrow \mathbb{R}$ is a continuous function. Therefore, Fredholm integral equation (5.3) has a solution. It can be easily seen that $u(\kappa)=\sin \left(\frac{\pi \kappa}{2}\right)$ is a solution of integral equation (5.3).

## 6. Numerical results

In this section, we show the effectiveness and efficiency of the algorithm (3.1) over SIMR. More precisely, we present a numerical example to settle our claims.

Example 6.1. Let $\mathcal{M}=\mathbb{R}$ endowed with the usual norm $\|\cdot\|$ and let $G, J$ : $\mathcal{M} \rightarrow \mathcal{M}$ be mappings defined as

$$
G(u)=L u, \quad L>0
$$

and $J(u)=u \quad$ for all $u \in \mathcal{M}$.
Let $\psi: \mathcal{M} \rightarrow \mathcal{M}$ be a contraction mapping defined as

$$
\psi(u)= \begin{cases}\frac{u}{2} & \text { if } G(u) \neq 0 \\ u & \text { if } G(u)=0\end{cases}
$$

Now,

$$
\|G(u)-G(v)\|=\|L u-L v\|=L\|u-v\| .
$$

Then $G$ is an $L$-Lipschitz continuous monotone, $J$ is 1 -Lipschitz continuous, $\psi$ is a contractive mapping with $\rho=\frac{1}{2}$, and $(G J)^{-1}(0)=0$.

We choose $\alpha_{n}=\frac{1}{(n+1)}, \beta_{n}=\frac{n-1}{(n+1)}, \gamma_{n}=\frac{1}{(n+1)}$, and $\omega_{n}=\frac{1}{n(n+1)}$. It can be seen that all these parameters satisfy conditions (H1) and (H2). The convergence behaviors of new algorithm (3.1) and SIMR are presented in Table 1 and Figures 1 and 2 below. We make different choices of coefficient $\left(s_{n}\right)$ and initial guesses. We set $\left\|u_{n+1}-u_{n}\right\|<10^{-5}$ as our stopping criterion.


Figure 1. Convergence behavior for initial guess $u_{1}=0.5$ and $L=3$.


Figure 2. Convergence behavior for initial guess $u_{1}=0.5$ and $L=20$.


Figure 3. Convergence behavior for initial guess $u_{1}=0.8$ and $L=0.9$.


Figure 4. Convergence behavior initial guess $u_{1}=0.8$ and $L=5$.


Figure 5. Convergence behavior for initial guess $u_{1}=1$ and $L=$ 0.25 .


Figure 6. Convergence behavior for initial guess $u_{1}=1$ and $L=$ 10.

TABLE 1. Influence of initial guess: Comparison of both iteration processes.

| For initial point $u_{1}=0.5$ and $L=3$ |  |
| :---: | :---: |
| Algorithm | Convergence in number of iterations |
| SIMA | 182 |
| New Algorithm (3.1) with $s_{n}=0.99$ | $\mathbf{1 5}$ |
| For initial point $u_{1}=0.5$ and $L=20$ |  |
| SIMA | 73 |
| New Algorithm (3.1) with $s_{n}=0.1$ | $\mathbf{2 6}$ |
| For initial point $u_{1}=0.8$ and $L=0.9$ |  |
| SIMA | 765 |
| New Algorithm (3.1) with $s_{n}=0.1$ | $\mathbf{5 7 3}$ |
| For initial point $u_{1}=0.8$ and $L=5$ |  |
| SIMA | 181 |
| New Algorithm (3.1) with $s_{n}=0.35$ | $\mathbf{3 6}$ |
| For initial point $u_{1}=1$ and $L=0.25$ |  |
| SIMA | 1141 |
| New Algorithm (3.1) with $s_{n}=0.01$ | $\mathbf{8 4 4}$ |
| For initial point $u_{1}=1$ and $L=10$ | 332 |
| SIMA | $\mathbf{1 6}$ |
| New Algorithm (3.1) with $s_{n}=0.13$ |  |

## Observations:

From Table 1 and Figures 1-6, we note that for different choices of initial guesses and parameters $L$, new algorithm (3.1) converges faster than SIMA.

Acknowledgement. We are very much thankful to the reviewers for their constructive comments and suggestions which have been useful for the improvement of this paper. The second author acknowledges with thanks the support from GES fellowship 4.0., University of Johannesburg, South Africa.

## References

1. R.P. Agarwal, M. Meehan, and D. O'Regan, Fixed Point Theory and Applications, Cambridge Tracts in Mathematics 141, Cambridge University Press, Cambridge, 2001.
2. Y.I Alber, Metric and generalized projection operators in Banach spaces: properties and applications, in: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, pp. 15-50, Lecture Notes in Pure and Appl. Math. 178, Dekker, New York, 1996.
3. W. Auzinger and R. Frank, Asymptotic error expansions for stiff equations: An analysis for the implicit midpoint and trapezoidal rules in the strongly stiff case, Numer. Math. 56 (1989), no. 5, 469-499.
4. M.S. Berger, Nonlinearity and Functional Analysis, Lectures on Nonlinear Problems in Mathematical Analysis, Academic Press, New York-London, 1977.
5. F.E. Browder, Fixed-point theorems for noncompact mappings in Hilbert space, Proc. Nat. Acad. Sci. U.S.A. 53 (1965) 1272-1276.
6. F.E. Browder, Nonlinear mappings of nonexpansive and accretive type in Banach spaces, Bull. Amer. Math. Soc. 73 (1967) 875-882.
7. F.E. Browder, Nonlinear operators and nonlinear equations of evolution in Banach spaces, in: Nonlinear Functional Analysis (Proc. Sympos. Pure Math. Vol. XVIII, Part 2, Chicago, Ill. 1968), pp. 1-308. Amer. Math. Soc. Providence, RI. 1976.
8. C. Chidume, Geometric Properties of Banach Spaces and Nonlinear Iterations, Lecture Notes in Math. 1965, Springer-Verlag, London, 2009.
9. C.E. Chidume and K.O. Idu, Approximation of zeros of bounded maximal monotone mappings, solutions of Hammerstein integral equations and convex minimization problems, Fixed Point Theory Appl. 2016 (2016), Paper No. 97, 28 pp.
10. T. Ibaraki and W. Takahashi, A new projection and convergence theorems for the projections in Banach spaces, J. Approx. Theory 149 (2007), no. 1, 1-14.
11. T. Kato, Nonlinear semigroups and evolution equations, J. Math. Soc. Japan 19 (1967) 508-520.
12. S. Khorasani and A. Adibi, Analytical solution of linear ordinary differential equations by differential transfer matrix method, Electron. J. Differential Equations 2003 (2003), No. 79, 18 pp .
13. G.J. Minty, Monotone (nonlinear) operators in Hilbert space, Duke Math. J. 29 (1962) 341-346.
14. A. Moudafi, Viscosity approximation methods for fixed-points problems, J. Math. Anal. Appl. 241 (2000), no. 1, 46-55.
15. R. Pant, R. Shukla and A. Petruşel, Viscosity approximation methods for generalized multivalued nonexpansive mappings with applications, Numer. Funct. Anal. Optim. 39 (2018), no. 13, 1374-1406.
16. N. Shioji and W. Takahashi, Strong convergence of approximated sequences for nonexpansive mappings in Banach spaces, Proc. Amer. Math. Soc. 125 (1997), no. 12, 3641-3645.
17. R. Shukla and R. Pant, Approximating solution of split equality and equilibrium problems by viscosity approximation algorithms, Comput. Appl. Math. 37 (2018), no. 4, 5293-5314.
18. R. Shukla and R. Pant, Robustness of theta method for nonexpansive mappings, Iran. J. Sci. Technol. Trans. A Sci. 43 (2019), no. 5, 2275-2284.
19. K.K. Tan and H.K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. 178 (1993), no. 2, 301-308.
20. Y. Tang and Z. Bao, New semi-implicit midpoint rule for zero of monotone mappings in Banach spaces, Numer. Algorithms 81 (2019), no. 3, 853-878.
21. H.K. Xu, An iterative approach to quadratic optimization, J. Optim. Theory Appl. 116 (2003), no. 3, 659-678.
22. H.K. Xu, M.A. Alghamdi and N. Shahzad, The viscosity technique for the implicit midpoint rule of nonexpansive mappings in Hilbert spaces, Fixed Point Theory Appl. 2015 (2015), Art. no. 41, 12 pp.
23. H. Zegeye, Strong convergence theorems for maximal monotone mappings in Banach spaces, J. Math. Anal. Appl. 343 (2008), no. 2, 663-671.
${ }^{1}$ University of the Gambia, Brikama Campus, Gambia
Email address: jt.mendy@utg.edu.gm
${ }^{2}$ Department of Mathematics \& Applied Mathematics, University of JohannesBURG
Kingsway Campus, Auckland Park 2006, South Africa
Email address: rshukla.vnit@gmail.com

[^0]:    Date: Received: 6 September 2020; Revised: 10 September 2020; Accepted: 5 October 2021.
    *Corresponding author.
    2020 Mathematics Subject Classification. Primary 47H10; Secondary 47H09, 47J05, 47J25, 54H25.

    Key words and phrases. Viscosity approximation method, Lipschitz mapping, Banach space.

