



# Khayyam Journal of Mathematics

emis.de/journals/KJM  
kjm-math.org

## $n$ -DUAL SPACES ASSOCIATED TO A NORMED SPACE

YOSAFAT E. P. PANGALELA

Communicated by M. Filali

**ABSTRACT.** For a real normed space  $X$ , we study the  $n$ -dual space of  $(X, \|\cdot\|)$  and show that the space is a Banach space. Meanwhile, for a real normed space  $X$  of dimension  $d \geq n$  which satisfies property (G), we discuss the  $n$ -dual space of  $(X, \|\cdot, \dots, \cdot\|_G)$ , where  $\|\cdot, \dots, \cdot\|_G$  is the Gähler  $n$ -norm. We then investigate the relationship between the  $n$ -dual space of  $(X, \|\cdot\|)$  and the  $n$ -dual space of  $(X, \|\cdot, \dots, \cdot\|_G)$ . We use this relationship to determine the  $n$ -dual space of  $(X, \|\cdot, \dots, \cdot\|_G)$  and show that the space is also a Banach space.

### 1. INTRODUCTION

In the 1960's, the notion of  $n$ -normed spaces was introduced by Gähler [2, 3, 4, 5] as a generalisation of normed spaces. For every real normed space  $X$  of dimension  $d \geq n$ , Gähler showed that  $X$  can be viewed as an  $n$ -normed space by using the Gähler  $n$ -norm, which is denoted by  $\|\cdot, \dots, \cdot\|_G$ . This  $n$ -norm is defined by using the set of bounded linear functionals on  $X$ . Since then, many researchers have studied operators and functionals on  $n$ -normed space  $X$  (see [1, 6, 9, 10, 11, 13, 14, 15]).

In [13], the author and Gunawan introduced the concept of  $n$ -dual spaces. For every real normed space  $X$  of dimension  $d \geq n$ , there are two  $n$ -dual spaces associated to  $X$ . The first is the  $n$ -dual space of  $(X, \|\cdot\|)$ , and the other is the  $n$ -dual space of  $(X, \|\cdot, \dots, \cdot\|_G)$ . In case  $X$  is the  $l^p$  space for some  $1 \leq p < \infty$ , the author and Gunawan have investigated and given the relationship between both  $n$ -dual spaces [13]. Here we provide an analogues result on more general normed spaces.

*Date:* Received: 16 August 2015, Accepted: 04 December 2015.

*2010 Mathematics Subject Classification.* Primary 46B20; Secondary 46C05, 46C15, 46B99, 46C99.

*Key words and phrases.*  $n$ -dual spaces,  $n$ -normed spaces, bounded linear functionals.

For a real normed space  $X$ , we investigate the  $n$ -dual space of  $(X, \|\cdot\|)$  by using the  $(n-1)$ -dual space of  $(X, \|\cdot\|)$  (Theorem 3.2). We then focus on a real normed space  $X$  of dimension  $d \geq n$  which satisfies property (G) and discuss the relationship between the  $n$ -dual space of  $(X, \|\cdot\|)$  and the  $n$ -dual space of  $(X, \|\cdot, \dots, \cdot\|_G)$  (Theorem 4.5). It is interesting to observe that both the  $n$ -dual space of  $(X, \|\cdot\|)$  and the  $n$ -dual space of  $(X, \|\cdot, \dots, \cdot\|_G)$  are Banach spaces (Theorem 3.3 and Theorem 4.7).

## 2. PRELIMINARIES

Let  $n$  be a nonnegative integer and  $X$  a real vector space of dimension  $d \geq n$ . We call a real-valued function  $\|\cdot, \dots, \cdot\|$  on  $X^n$  an  $n$ -norm on  $X$  if for all  $x_1, \dots, x_n, x' \in X$ , we have

- (1)  $\|x_1, \dots, x_n\| = 0$  if and only if  $x_1, \dots, x_n$  are linearly dependent;
- (2)  $\|x_1, \dots, x_n\|$  is invariant under permutation;
- (3)  $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \|x_1, x_2, \dots, x_n\|$  for all  $\alpha \in \mathbb{R}$ ; and
- (4)  $\|x_1 + x', \dots, x_n\| \leq \|x_1, \dots, x_n\| + \|x', \dots, x_n\|$ .

We then call the pair  $(X, \|\cdot, \dots, \cdot\|)$  an  $n$ -normed space.

An example of an  $n$ -normed space is the  $l^p$  space, where  $1 \leq p < \infty$ , equipped with

$$\|x_1, \dots, x_n\|_p := \left( \frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} |\det(x_{ij_k})|^p \right)^{\frac{1}{p}}$$

for  $x_1, \dots, x_n \in l^p$  (see [7, Section 3]).

Another interesting example of  $n$ -normed spaces is the Gähler  $n$ -norm which was introduced in [3, 4, 5]. Let  $X$  be a real normed space of dimension  $d \geq n$ , and  $X^{(1)}$  the dual space of  $X$ . Gähler showed that the function  $\|\cdot, \dots, \cdot\|_G$  which is given by

$$\|x_1, \dots, x_n\|_G := \sup_{\substack{f_i \in X^{(1)}, \|f_i\| \leq 1 \\ 1 \leq i \leq n}} \left| \det [f_j(x_i)]_{i,j} \right|$$

for all  $x_1, \dots, x_n \in X$ , is an  $n$ -norm on  $X$ . Hence every real normed space  $X$  can be viewed as an  $n$ -normed space  $(X, \|\cdot, \dots, \cdot\|_G)$ .

Let  $X$  be a real normed space of dimension  $d \geq n$ . Any real-valued function  $f$  on  $X^n$  is called an  $n$ -functional on  $X$ . An  $n$ -functional  $f$  is *multilinear* if it satisfies two following properties:

- (1)  $f(x_1 + y_1, \dots, x_n + y_n) = \sum_{z_i \in \{x_i, y_i\}, 1 \leq i \leq n} f(z_1, \dots, z_n)$  and
- (2)  $f(\alpha_1 x_1, \dots, \alpha_n x_n) = \alpha_1 \cdots \alpha_n f(x_1, \dots, x_n)$

for all  $x_1, \dots, x_n, y_1, \dots, y_n \in X$  and  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ .

For multilinear  $n$ -functionals  $f, h$  on  $X$ , we define an  $n$ -functional  $f + h$  by

$$(f + h)(x_1, \dots, x_n) := f(x_1, \dots, x_n) + h(x_1, \dots, x_n)$$

for  $x_1, \dots, x_n \in X$ . Then  $f + h$  is also multilinear. On the other hand, we say  $f = h$  if

$$f(x_1, \dots, x_n) = h(x_1, \dots, x_n)$$

for  $x_1, \dots, x_n \in X$ .

We call an  $n$ -functional  $f$  *bounded* on a real normed space  $(X, \|\cdot\|)$  (respectively, an  $n$ -normed space  $(X, \|\cdot, \dots, \cdot\|)$ ) if there exists a constant  $K > 0$  such that

$$|f(x_1, \dots, x_n)| \leq K \|x_1\| \cdots \|x_n\| \quad (\text{respectively, } |f(x_1, \dots, x_n)| \leq K \|x_1, \dots, x_n\|)$$

for all  $x_1, \dots, x_n \in X$ .

Let  $S_n$  denote the group of permutations of  $(1, \dots, n)$ . Recall from [13] that every bounded multilinear  $n$ -functional  $f$  on  $(X, \|\cdot, \dots, \cdot\|)$  is *antisymmetric* in the sense that

$$f(x_1, \dots, x_n) = \text{sgn}(\sigma) f(x_{\sigma(1)}, \dots, x_{\sigma(n)})$$

for  $x_1, \dots, x_n \in X$  and  $\sigma \in S_n$ . Here  $\text{sgn}(\sigma) = 1$  if  $\sigma$  is an even permutation, and  $\text{sgn}(\sigma) = -1$  if  $\sigma$  is an odd permutation. Note that if  $f$  is antisymmetric then for any linearly dependent  $x_1, \dots, x_n \in X$ , we have  $f(x_1, \dots, x_n) = 0$ .

*Remark 2.1.* In general, we do not have the antisymmetric property for bounded multilinear  $n$ -functionals on  $(X, \|\cdot\|)$ .

The space of bounded multilinear  $n$ -functionals on  $(X, \|\cdot\|)$  is called the  *$n$ -dual space* of  $(X, \|\cdot\|)$  and denoted by  $X^{(n)}$ . For  $n = 0$ , we define  $X^{(0)}$  as  $\mathbb{R}$ . The function  $\|\cdot\|_{n,1}$  on  $X^{(n)}$  where

$$\|f\|_{n,1} := \sup_{x_1, \dots, x_n \neq 0} \frac{|f(x_1, \dots, x_n)|}{\|x_1\| \cdots \|x_n\|}$$

for  $f \in X^{(n)}$ , defines a norm on  $X^{(n)}$  and then  $X^{(n)}$  is a normed space.

Meanwhile, the  *$n$ -dual space* of  $(X, \|\cdot, \dots, \cdot\|)$  is the space of all bounded multilinear  $n$ -functionals on  $(X, \|\cdot, \dots, \cdot\|)$ . This space is also a normed space with the following norm

$$\|f\|_{n,n} := \sup_{\|x_1, \dots, x_n\| \neq 0} \frac{|f(x_1, \dots, x_n)|}{\|x_1, \dots, x_n\|}.$$

Now let  $X, Y$  be real normed spaces. We write  $B(X, Y)$  to denote the set of bounded linear operators from  $X$  into  $Y$ . The function  $\|\cdot\|_{\text{op}}$  where

$$\|u\|_{\text{op}} := \sup_{x \neq 0} \frac{\|u(x)\|}{\|x\|}$$

for every  $u \in B(X, Y)$ , is a norm on  $B(X, Y)$ . For simplification, we write  $B(X, Y)$  to denote the normed space  $B(X, Y)$  equipped with  $\|\cdot\|_{\text{op}}$ . Otherwise, if  $\|\cdot\|^*$  is a norm function on  $B(X, Y)$ , we write  $(B(X, Y), \|\cdot\|^*)$  to denote the normed space  $B(X, Y)$  equipped by the norm  $\|\cdot\|^*$ .

### 3. THE $n$ -DUAL SPACE OF $(X, \|\cdot\|)$

In this section, we first identify the bounded multilinear  $n$ -functionals on  $(X, \|\cdot\|)$  (Proposition 3.1). We then identify the  $n$ -dual space of  $(X, \|\cdot\|)$  by using the  $(n - 1)$ -dual space of  $(X, \|\cdot\|)$  (Theorem 3.2). Finally we show that the  $n$ -dual space of  $(X, \|\cdot\|)$  is a Banach space (Theorem 3.3).

**Proposition 3.1.** *Let  $X$  be a real normed space of dimension  $d \geq n$  and  $f$  a bounded multilinear  $n$ -functional on  $(X, \|\cdot\|)$ . Then there exists  $u_f \in B(X, X^{(n-1)})$  such that for  $x_1, \dots, x_{n-1}, z \in X$ ,*

$$f(x_1, \dots, x_{n-1}, z) = (u_f(z))(x_1, \dots, x_{n-1}).$$

Furthermore,  $\|f\|_{n,1} = \|u_f\|_{\text{op}}$ .

*Proof.* Take  $z \in X$  and define an  $(n-1)$ -functional  $f_z$  on  $X$  with

$$f_z(x_1, \dots, x_{n-1}) := f(x_1, \dots, x_{n-1}, z)$$

for  $x_1, \dots, x_{n-1} \in X$ . We show  $f_z \in X^{(n-1)}$ . Note that for  $x_1, \dots, x_{n-1}, y_1, \dots, y_{n-1} \in X$  and  $\alpha_1, \dots, \alpha_{n-1} \in \mathbb{R}$ , we have

$$\begin{aligned} f_z(x_1 + y_1, \dots, x_{n-1} + y_{n-1}) &= f(x_1 + y_1, \dots, x_{n-1} + y_{n-1}, z) \\ &= \sum_{z_i \in \{x_i, y_i\}, 1 \leq i \leq n-1} f(z_1, \dots, z_{n-1}, z) \\ &= \sum_{z_i \in \{x_i, y_i\}, 1 \leq i \leq n-1} f_z(z_1, \dots, z_{n-1}), \end{aligned}$$

$$\begin{aligned} f_z(\alpha_1 x_1, \dots, \alpha_{n-1} x_{n-1}) &= f(\alpha_1 x_1, \dots, \alpha_{n-1} x_{n-1}, z) \\ &= \alpha_1 \cdots \alpha_{n-1} f(x_1, \dots, x_{n-1}, z) \\ &= \alpha_1 \cdots \alpha_{n-1} f_z(x_1, \dots, x_{n-1}), \end{aligned}$$

and

$$|f_z(x_1, \dots, x_{n-1})| = |f(x_1, \dots, x_{n-1}, z)| \leq \|f\|_{n,1} \|z\| (\|x_1\| \cdots \|x_{n-1}\|)$$

since  $f$  is bounded on  $(X, \|\cdot\|)$ . Hence  $f_z : X^{n-1} \rightarrow \mathbb{R}$  is multilinear and bounded; and then  $f_z \in X^{(n-1)}$ .

Now define  $u_f : X \rightarrow X^{(n-1)}$  with  $u_f(z) := f_z$  for  $z \in X$ . We have to show  $u_f \in B(X, X^{(n-1)})$ . First we show that  $u_f$  is linear. Take  $z_1, z_2 \in X$  and  $\alpha, \beta \in \mathbb{R}$ . For every  $x_1, \dots, x_{n-1} \in X$ , we have

$$\begin{aligned} (u_f(\alpha z_1 + \beta z_2))(x_1, \dots, x_{n-1}) &= f_{\alpha z_1 + \beta z_2}(x_1, \dots, x_{n-1}) \\ &= f(x_1, \dots, x_{n-1}, \alpha z_1 + \beta z_2) \\ &= f(x_1, \dots, x_{n-1}, \alpha z_1) + f(x_1, \dots, x_{n-1}, \beta z_2) \\ &= \alpha f(x_1, \dots, x_{n-1}, z_1) + \beta f(x_1, \dots, x_{n-1}, z_2) \\ &= \alpha f_{z_1}(x_1, \dots, x_{n-1}) + \beta f_{z_2}(x_1, \dots, x_{n-1}) \\ &= (\alpha u_f(z_1))(x_1, \dots, x_{n-1}) + (\beta u_f(z_2))(x_1, \dots, x_{n-1}) \\ &= (\alpha u_f(z_1) + \beta u_f(z_2))(x_1, \dots, x_{n-1}) \end{aligned}$$

and

$$u_f(\alpha z_1 + \beta z_2) = \alpha u_f(z_1) + \beta u_f(z_2).$$

Hence  $u_f$  is linear.

Next we show the boundedness of  $u_f$ . Take  $z \in X$ . Then for  $x_1, \dots, x_{n-1} \in X$ , we have

$$\begin{aligned} |(u_f(z))(x_1, \dots, x_{n-1})| &= |f_z(x_1, \dots, x_{n-1})| = |f(x_1, \dots, x_{n-1}, z)| \\ &\leq \|f\|_{n,1} \|x_1\| \cdots \|x_{n-1}\| \|z\| \quad (f \text{ is bounded on } (X, \|\cdot\|)) \end{aligned}$$

and then

$$\|u_f(z)\| = \sup_{x_1, \dots, x_{n-1} \neq 0} \frac{|(u_f(z))(x_1, \dots, x_{n-1})|}{\|x_1\| \cdots \|x_{n-1}\|} \leq \|f\|_{n,1} \|z\|$$

which is finite. This implies

$$\sup_{z \neq 0} \frac{\|u_f(z)\|}{\|z\|} \leq \|f\|_{n,1}$$

which is finite. Therefore  $u_f$  is bounded and  $\|u_f\|_{\text{op}} \leq \|f\|_{n,1}$ .

Finally we claim that  $\|u_f\|_{\text{op}} = \|f\|_{n,1}$ . Recall that we already have  $\|u_f\|_{\text{op}} \leq \|f\|_{n,1}$ . To show the reverse inequality, note that for  $z \in X$ ,  $u_f(z) = f_z$  is bounded. Then for  $x_1, \dots, x_{n-1}, z \in X$ ,

$$\begin{aligned} |f(x_1, \dots, x_{n-1}, z)| &= |f_z(x_1, \dots, x_{n-1})| \\ &\leq \|f_z\|_{n,1} \|x_1\| \cdots \|x_{n-1}\| \\ &\quad (f \text{ is bounded on } (X, \|\cdot\|)) \\ &= \|u_f(z)\| \|x_1\| \cdots \|x_{n-1}\| \\ &\leq (\|u_f\|_{\text{op}} \|z\|) \|x_1\| \cdots \|x_{n-1}\| \end{aligned}$$

since  $u_f$  is bounded. Hence

$$\|f\|_{n,1} = \sup_{x_1, \dots, x_{n-1}, z \neq 0} \frac{|f(x_1, \dots, x_{n-1}, z)|}{\|x_1\| \cdots \|x_{n-1}\| \|z\|} \leq \|u_f\|_{\text{op}}$$

and  $\|f\|_{n,1} \leq \|u_f\|_{\text{op}}$ . Therefore  $\|u_f\|_{\text{op}} = \|f\|_{n,1}$ , as claimed.  $\square$

**Theorem 3.2.** *Let  $X$  be a real normed space of dimension  $d \geq n$ . Then the  $n$ -dual space of  $(X, \|\cdot\|)$  is  $B(X, X^{(n-1)})$ .*

*Proof.* For a bounded multilinear  $n$ -functional  $f$  on  $(X, \|\cdot\|)$ , let  $u_f \in B(X, X^{(n-1)})$  be as in Proposition 3.1. Define a map  $\theta$  from the  $n$ -dual space of  $(X, \|\cdot\|)$  to  $B(X, X^{(n-1)})$  with

$$\theta(f) := u_f$$

for  $f \in X^{(n)}$ . We have to show that  $\theta$  is isometric and bijective.

The isometricness of  $\theta$  follows from Proposition 3.1.

Next we show the injectivity of  $\theta$ . Let  $f, h$  be bounded multilinear  $n$ -functionals on  $(X, \|\cdot\|)$  such that  $\theta(f) = \theta(h)$ . Then  $u_f = u_h$  and for every  $x_1, \dots, x_{n-1}, x_n \in X$ , we have

$$\begin{aligned} f(x_1, \dots, x_{n-1}, x_n) &= (u_f(x_n))(x_1, \dots, x_{n-1}) \\ &= (u_h(x_n))(x_1, \dots, x_{n-1}) \\ &= h(x_1, \dots, x_{n-1}, x_n). \end{aligned}$$

Hence  $f = h$  and  $\theta$  is injective.

To show that  $\theta$  is surjective, we take  $u \in B(X, X^{(n-1)})$  and have to show that there exists a bounded multilinear  $n$ -functional  $f_u$  on  $(X, \|\cdot\|)$  such that  $\theta(f_u) = u$ . Now we define  $f_u$  an  $n$ -functional on  $X$  where

$$f_u(x_1, \dots, x_{n-1}, x_n) := (u(x_n))(x_1, \dots, x_{n-1})$$

for  $x_1, \dots, x_{n-1}, x_n \in X$ . We claim that  $f_u$  is multilinear and bounded on  $(X, \|\cdot\|)$ .

First we show that  $f_u$  is multilinear. Take  $x_1, \dots, x_n, y_1, \dots, y_n \in X$  and  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ . We have

$$\begin{aligned} f_u(x_1 + y_1, \dots, x_n + y_n) &= (u(x_n + y_n))(x_1 + y_1, \dots, x_{n-1} + y_{n-1}) \\ &= \sum_{z_i \in \{x_i, y_i\}, 1 \leq i \leq n-1} (u(x_n + y_n))(z_1, \dots, z_{n-1}) \\ &= \sum_{z_i \in \{x_i, y_i\}, 1 \leq i \leq n-1} (u(x_n) + u(y_n))(z_1, \dots, z_{n-1}) \\ &= \sum_{z_i \in \{x_i, y_i\}, 1 \leq i \leq n-1} (f(z_1, \dots, z_{n-1}, x_n) + f(z_1, \dots, z_{n-1}, y_n)) \\ &= \sum_{z_i \in \{x_i, y_i\}, 1 \leq i \leq n} f(z_1, \dots, z_{n-1}, z_n) \end{aligned}$$

and

$$\begin{aligned} f_u(\alpha_1 x_1, \dots, \alpha_n x_n) &= (u(\alpha_n x_n))(\alpha_1 x_1, \dots, \alpha_{n-1} x_{n-1}) \\ &= \alpha_1 \cdots \alpha_{n-1} (u(\alpha_n x_n))(x_1, \dots, x_{n-1}) \quad (u(\alpha_n x_n) \text{ is multilinear}) \\ &= \alpha_1 \cdots \alpha_{n-1} \alpha_n (u(x_n))(x_1, \dots, x_{n-1}) \quad (u \text{ is linear}) \\ &= \alpha_1 \cdots \alpha_{n-1} \alpha_n f_u(x_1, \dots, x_{n-1}, x_n). \end{aligned}$$

Hence  $f_u$  is multilinear.

Next we show that  $f_u$  is bounded on  $(X, \|\cdot\|)$ . Take  $x_1, \dots, x_n \in X$ . Then

$$\begin{aligned} |f_u(x_1, \dots, x_{n-1}, x_n)| &= |(u(x_n))(x_1, \dots, x_{n-1})| \\ &\leq \|u(x_n)\| \|x_1\| \cdots \|x_{n-1}\| \quad (u(x_n) \text{ is bounded}) \\ &\leq (\|u\|_{\text{op}} \|x_n\|) \|x_1\| \cdots \|x_{n-1}\| \quad (u \text{ is bounded}) \end{aligned}$$

and  $f_u$  is bounded.

Hence  $f_u$  is multilinear and bounded on  $(X, \|\cdot\|)$ , as claimed. Note that  $\theta(f_u) = u_{f_u}$ . Take  $x_1, \dots, x_n \in X$  and we have

$$(u(x_n))(x_1, \dots, x_{n-1}) = f_u(x_1, \dots, x_{n-1}, x_n) = ((u_{f_u})(x_n))(x_1, \dots, x_{n-1}).$$

Then  $u(x_n) = u_{f_u}(x_n)$  for  $x_n \in X$ , and

$$u = u_{f_u} = \theta(f_u).$$

Therefore,  $\theta$  is surjective and a bijection, as required.  $\square$

Recall from [8, Theorem 2.10-2] that for normed spaces  $X, Y$ , the normed space  $B(X, Y)$  is a Banach space if  $Y$  is a Banach space. Since  $\mathbb{R}$  is a Banach space, then for every normed space  $X$ ,  $X^{(1)}$  is also a Banach space. Hence Theorem 3.2 with  $n = 2$  implies that  $X^{(2)}$  is also a Banach space. Therefore, by induction and Theorem 3.2, we get the following theorem.

**Theorem 3.3.** *Let  $X$  be a real normed space of dimension  $d \geq n$ . Then the  $n$ -dual space of  $(X, \|\cdot\|)$  is a Banach space.*

#### 4. THE $n$ -DUAL SPACE OF $(X, \|\cdot, \dots, \cdot\|_G)$

In this section, we focus on normed spaces of dimension  $d \geq n$  which satisfy property (G). On this space, we investigate the relationship between bounded multilinear  $n$ -functionals on  $(X, \|\cdot, \dots, \cdot\|_G)$  and bounded multilinear  $n$ -functionals on  $(X, \|\cdot\|)$

(Lemma 4.3). We then use it to determine the  $n$ -dual space of  $(X, \|\cdot, \dots, \cdot\|_G)$  (Theorem 4.5) and show that the space is a Banach space (Theorem 4.7).

First we recall the functional  $g$  and property (G) introduced by Miličić in [12]. The functional  $g : X^2 \rightarrow \mathbb{R}$  is defined by

$$g(x, y) := \frac{\|x\|}{2} (\tau_-(x, y) + \tau_+(x, y))$$

where

$$\tau_{\pm}(x, y) := \lim_{t \rightarrow \pm 0} t^{-1} (\|x + ty\| - \|x\|).$$

The functional  $g$  satisfies the following properties: for all  $x, y \in X$  and  $\alpha, \beta \in \mathbb{R}$

- (G1)  $g(x, x) = \|x\|^2$ ;
- (G2)  $g(\alpha x, \beta y) = \alpha\beta g(x, y)$ ;
- (G3)  $g(x, x + y) = \|x\|^2 + g(x, y)$ ; and
- (G4)  $|g(x, y)| \leq \|x\| \|y\|$ .

We say that a real normed space  $X$  satisfies *property (G)* if the functional  $g(x, y)$  is linear with respect to  $y \in X$ . In that case, we then call  $g$  a *semi-inner product* on  $X$ . For example, for  $1 \leq p < \infty$ , the  $l^p$  space satisfies property (G) (see [16]).

By using the semi-inner product  $g$ , we define an orthogonal relation on  $X$  as follows:

$$x \perp_g y \Leftrightarrow g(x, y) = 0.$$

Let  $x \in X$  and  $Y = \{y_1, \dots, y_n\} \subseteq X$ . We write  $\Gamma(y_1, \dots, y_n)$  to denote the Gram determinant  $\det [g(y_i, y_j)]_{i,j}$ . If  $\Gamma(y_1, \dots, y_n) \neq 0$ , then the vector

$$x_Y := -\frac{1}{\Gamma(y_1, \dots, y_n)} \det \begin{bmatrix} 0 & y_1 & \cdots & y_n \\ g(y_1, x) & g(y_1, y_1) & \cdots & g(y_1, y_n) \\ \vdots & \vdots & & \vdots \\ g(y_n, x) & g(y_n, y_1) & \cdots & g(y_n, y_n) \end{bmatrix}$$

is called the Gram-Schmidt projection of the vector  $x$  on  $Y$ .

Next let  $\{x_1, \dots, x_n\}$  be a linearly independent set of vectors in  $X$ . As in [12], we call  $x_1^\circ, \dots, x_n^\circ$  the *left  $g$ -orthogonal sequence* where  $x_1^\circ := x_1$  and for  $i = 2, \dots, n$ ,

$$x_i^\circ := x_i - (x_i)_{S_{i-1}},$$

where  $S_{i-1} := \text{span}\{x_1, \dots, x_{i-1}\}$ . Note that if  $i < j$ , then  $x_i^\circ \perp_g x_j^\circ$  and  $g(x_i^\circ, x_j^\circ) = 0$ .

**Proposition 4.1.** *Let  $X$  be a real normed space of dimension  $d \geq n$  which satisfies property (G). Let  $\{x_1, \dots, x_n\}$  be a linearly independent set of vectors in  $X$ . Then*

$$\|x_1^\circ\| \cdots \|x_n^\circ\| \leq \|x_1, \dots, x_n\|_G \leq n! \|x_1\| \cdots \|x_n\|.$$

*Proof.* First we show the right inequality. Note that

$$\begin{aligned}
\|x_1, \dots, x_n\|_G &= \sup_{\substack{f_i \in X^{(1)}, \|f_i\| \leq 1 \\ 1 \leq i \leq n}} \left| \det [f_j(x_i)]_{i,j} \right| \\
&= \sup_{\substack{f_i \in X^{(1)}, \|f_i\| \leq 1 \\ 1 \leq i \leq n}} \left| \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^n f_{\sigma(i)}(x_i) \right| \quad (\text{by the Leibniz formula}) \\
&\leq \sup_{\substack{f_i \in X^{(1)}, \|f_i\| \leq 1 \\ 1 \leq i \leq n}} \sum_{\sigma \in S_n} \left| \prod_{i=1}^n f_{\sigma(i)}(x_i) \right| \quad (\text{by the triangle inequality}) \\
&\leq \sup_{\substack{f_i \in X^{(1)}, \|f_i\| \leq 1 \\ 1 \leq i \leq n}} \sum_{\sigma \in S_n} \left( \prod_{i=1}^n \|f_{\sigma(i)}\| \|x_i\| \right) \quad (\text{each } f_i \text{ is bounded}) \\
&\leq \sum_{\sigma \in S_n} \left( \prod_{i=1}^n \|x_i\| \right) \\
&= n! \|x_1\| \cdots \|x_n\|,
\end{aligned}$$

as required.

To show the left inequality, we first show that for a fixed  $x \in X$ , the functional  $g_x$  on  $X$  defined by

$$g_x(y) := \frac{g(x, y)}{\|x\|}$$

for  $y \in X$ , is bounded and linear. The linearity follows since  $X$  satisfies property (G). Now take  $y \in X$ , by (G4), we have

$$|g_x(y)| = \left| \frac{g(x, y)}{\|x\|} \right| \leq \|y\|$$

and  $g_x$  is bounded, as required. Hence for  $x \in X$ ,  $g_x \in X^{(1)}$ . Furthermore,  $\|g_x\| \leq 1$ .

Now note that  $\|x_1, \dots, x_n\|_G = \|x_1^\circ, \dots, x_n^\circ\|_G$ . This implies

$$\begin{aligned}
\|x_1, \dots, x_n\|_G &= \|x_1^\circ, \dots, x_n^\circ\|_G = \sup_{\substack{f_i \in X^{(1)}, \|f_i\| \leq 1 \\ 1 \leq i \leq n}} \left| \det [f_j(x_i^\circ)]_{i,j} \right| \quad (4.1) \\
&\geq \left| \det [g_{x_j^\circ}(x_i^\circ)]_{i,j} \right| = \frac{1}{\|x_1^\circ\| \cdots \|x_n^\circ\|} \left| \det [g(x_j^\circ, x_i^\circ)]_{i,j} \right|.
\end{aligned}$$

Since  $x_1^\circ, \dots, x_n^\circ$  is the left  $g$ -orthogonal sequence, then  $g(x_i^\circ, x_j^\circ) = 0$  if  $i < j$ . By (G1), we get  $g(x_i^\circ, x_i^\circ) = \|x_i^\circ\|^2$  for  $i = 1, \dots, n$ . This implies

$$\left| \det [g(x_j^\circ, x_i^\circ)]_{i,j} \right| = \|x_1^\circ\|^2 \cdots \|x_n^\circ\|^2$$

and (4.1) become

$$\|x_1, \dots, x_n\|_G \geq \|x_1^\circ\| \cdots \|x_n^\circ\|,$$

as required.  $\square$

*Remark 4.2.* Proposition 4.1 is a generalisation of Theorem 2.2 in [16]. In [16, Theorem 2.2], Wibawa-Kusumah and Gunawan only proved Proposition 4.1 for  $l^p$  spaces where  $1 \leq p < \infty$ .



**Lemma 4.3.** *Let  $X$  be a real normed space of dimension  $d \geq n$  which satisfies property (G). Let  $f$  be a multilinear  $n$ -functional on  $X$ . Then  $f$  is antisymmetric and bounded on  $(X, \|\cdot\|)$  if and only if  $f$  is bounded on  $(X, \|\cdot, \dots, \cdot\|_G)$ . Furthermore*

$$\|f\|_{n,n} \leq \|f\|_{n,1} \leq n! \|f\|_{n,n}.$$

*Proof.* First suppose that  $f$  is antisymmetric bounded on  $(X, \|\cdot\|)$ . Take linearly independent  $x_1, \dots, x_n \in X$ . Then

$$f(x_1, \dots, x_n) = f(x_1^\circ, \dots, x_n^\circ)$$

and by the left inequality in Proposition 4.1,

$$\begin{aligned} \frac{|f(x_1, \dots, x_n)|}{\|x_1, \dots, x_n\|_G} &\leq \frac{|f(x_1, \dots, x_n)|}{\|x_1^\circ\| \cdots \|x_n^\circ\|} = \frac{|f(x_1^\circ, \dots, x_n^\circ)|}{\|x_1^\circ\| \cdots \|x_n^\circ\|} \\ &\leq \|f\|_{n,1} \quad (f \text{ is bounded on } (X, \|\cdot\|)) \end{aligned}$$

which is finite. Hence  $f$  is bounded on  $(X, \|\cdot, \dots, \cdot\|_G)$  and

$$\|f\|_{n,n} \leq \|f\|_{n,1}. \tag{4.2}$$

Next suppose that  $f$  is bounded on  $(X, \|\cdot, \dots, \cdot\|_G)$ . Then  $f$  is antisymmetric. To show the boundedness of  $f$  on  $(X, \|\cdot\|)$ , we take linearly independent  $x_1, \dots, x_n \in X$ . Then by the right inequality in Proposition 4.1,

$$\begin{aligned} \frac{|f(x_1, \dots, x_n)|}{\|x_1\| \cdots \|x_n\|} &\leq n! \frac{|f(x_1, \dots, x_n)|}{\|x_1, \dots, x_n\|_G} \\ &\leq n! \|f\|_{n,n} \quad (f \text{ is bounded on } (X, \|\cdot, \dots, \cdot\|_G)) \end{aligned}$$

which is finite. Hence  $f$  is bounded on  $(X, \|\cdot\|)$  and

$$\|f\|_{n,1} \leq n! \|f\|_{n,n}. \tag{4.3}$$

Finally, by (4.2) and (4.3), we get

$$\|f\|_{n,n} \leq \|f\|_{n,1} \leq n! \|f\|_{n,n},$$

as required. □

Now we say  $u \in B(X, X^{(n-1)})$  *antisymmetric* if for  $x_1, \dots, x_n \in X$  and  $\sigma \in S_n$ ,

$$(u(x_n))(x_1, \dots, x_{n-1}) = \text{sgn}(\sigma) (u(x_{\sigma(n)}))(x_{\sigma(1)}, \dots, x_{\sigma(n-1)})$$

and then define  $B_{\text{as}}(X, X^{(n-1)})$  as the collection of antisymmetric elements of  $B(X, X^{(n-1)})$ . Note that  $B_{\text{as}}(X, X^{(n-1)})$  is also a normed space with the norm inherited from  $B(X, X^{(n-1)})$  which is  $\|\cdot\|_{\text{op}}$ .

Note that Theorem 3.2 and Lemma 4.3 imply that every bounded multilinear  $n$ -functional on  $(X, \|\cdot, \dots, \cdot\|_G)$  can be identified as an element of  $B_{\text{as}}(X, X^{(n-1)})$  and vice versa. Therefore Lemma 4.3 implies the following corollary and theorem.

**Corollary 4.4.** *Let  $X$  be a real normed space of dimension  $d \geq n$  which satisfies property (G). The function  $\|\cdot\|_G$  on  $B_{\text{as}}(X, X^{(n-1)})$  where*

$$\|u\|_G := \sup_{\|x_1, \dots, x_n\|_G \neq 0} \frac{|(u(x_n))(x_1, \dots, x_{n-1})|}{\|x_1, \dots, x_n\|_G}$$

*for  $u \in B(X, X^{(n-1)})$ , defines a norm on  $B_{\text{as}}(X, X^{(n-1)})$ . Furthermore,  $\|\cdot\|_G$  and  $\|\cdot\|_{\text{op}}$  are equivalent norms on  $B_{\text{as}}(X, X^{(n-1)})$  with*

$$\|u\|_G \leq \|u\|_{\text{op}} \leq n! \|u\|_G$$

for  $u \in B(X, X^{(n-1)})$ .

**Theorem 4.5.** *Let  $X$  be a real normed space of dimension  $d \geq n$  which satisfies property (G). Then the  $n$ -dual space of  $(X, \|\cdot, \dots, \cdot\|_G)$  is  $(B_{\text{as}}(X, X^{(n-1)}), \|\cdot\|_G)$ .*

The rest of this section is devoted to show that for  $n \in \mathbb{N}$ , the  $n$ -dual space of  $(X, \|\cdot, \dots, \cdot\|_G)$  is a Banach space.

**Theorem 4.6.** *Let  $X$  be a real normed space of dimension  $d \geq n$  which satisfies property (G). Then  $B_{\text{as}}(X, X^{(n-1)})$  is a Banach space.*

*Proof.* Since every closed subspace of a Banach space is also a Banach space, then by Theorem 3.3, it suffices to show that  $B_{\text{as}}(X, X^{(n-1)})$  is a closed subspace of  $B(X, X^{(n-1)})$ .

Take a sequence  $\{u_m\} \subseteq B_{\text{as}}(X, X^{(n-1)})$  such that  $u_m \rightarrow u$ . We have to show  $u \in B_{\text{as}}(X, X^{(n-1)})$ . In other words, for  $x_1, \dots, x_n \in X$  and  $\sigma \in S_n$ , we have to show

$$(u(x_n))(x_1, \dots, x_{n-1}) = \text{sgn}(\sigma) (u(x_{\sigma(n)}))(x_{\sigma(1)}, \dots, x_{\sigma(n-1)}).$$

Take  $x_1, \dots, x_n \in X$  and  $\sigma \in S_n$ . First note that for  $m \in \mathbb{N}$ , we have

$$\|u(x_n) - u_m(x_n)\| = \|(u - u_m)(x_n)\| \leq \|u - u_m\|_{\text{op}} \|x_n\| \quad (4.4)$$

since  $u - u_m$  is bounded. Since  $u(x_n), u_m(x_n) \in X^{(n-1)}$ , then  $(u - u_m)(x_n)$  is bounded and for  $y_1, \dots, y_{n-1} \in X$ , we have

$$\|((u - u_m)(x_n))(y_1, \dots, y_{n-1})\| \leq \|u(x_n) - u_m(x_n)\| \|y_1\| \cdots \|y_{n-1}\|. \quad (4.5)$$

Since  $u_m \rightarrow u$ , then by (4.4) and (4.5), we get

$$(u_m(x_n))(y_1, \dots, y_{n-1}) \rightarrow (u(x_n))(y_1, \dots, y_{n-1}) \quad (4.6)$$

for  $y_1, \dots, y_{n-1} \in X$ . Since  $u_m$  is antisymmetric for every  $m \in \mathbb{N}$ , then (4.6) implies

$$(u(x_n))(x_1, \dots, x_{n-1}) = \text{sgn}(\sigma) (u(x_{\sigma(n)}))(x_{\sigma(1)}, \dots, x_{\sigma(n-1)}),$$

as required. Thus  $B_{\text{as}}(X, X^{(n-1)})$  is closed and then a Banach space.  $\square$

Furthermore, since  $\|\cdot\|_G$  and  $\|\cdot\|_{\text{op}}$  are equivalent norms on  $B_{\text{as}}(X, X^{(n-1)})$ , then by Theorem 4.5 and Theorem 4.6, we get the following theorem.

**Theorem 4.7.** *Let  $X$  be a real normed space of dimension  $d \geq n$  which satisfies property (G). Then the  $n$ -dual space of  $(X, \|\cdot, \dots, \cdot\|_G)$  is a Banach space.*

#### REFERENCES

1. H. Batkude, H. Gunawan, and Y.E.P. Pangalela, *Bounded linear functionals on the  $n$ -normed space of  $p$ -summable sequences*, Acta Univ. M. Belii Ser. Math. **21** (2013), 66–75.
2. S. Gähler, *Lineare 2-normierte räume*, Math. Nachr. **28** (1964), 1–43.
3. S. Gähler, *Untersuchungen über verallgemeinerte  $m$ -metrische räume I*, Math. Nachr. **40** (1969), 165–189.
4. S. Gähler, *Untersuchungen über verallgemeinerte  $m$ -metrische räume II*, Math. Nachr. **40** (1969), 229–264.
5. S. Gähler, *Untersuchungen über verallgemeinerte  $m$ -metrische räume III*, Math. Nachr. **41** (1969), 23–36.
6. S.M. Gozali, H. Gunawan and O. Neswan, *On  $n$ -norms and  $n$ -bounded linear functionals in a Hilbert space*, Ann. Funct. Anal. **1** (2010), 72–79.
7. H. Gunawan, *The space of  $p$ -summable sequences and its natural  $n$ -norm*, Bull. Aust. Math. Soc. **64** (2001), 137–147.
8. E. Kreyszig, *Introductory Functional Analysis with Applications*, John Wiley and Sons, Inc. 1978.

9. S.N. Lal and M. Das, *2-functionals and some extension theorems in linear spaces*, Indian J. Pure Appl. Math. **13** (1982), 912–919.
10. Z. Lewandowska, *Bounded 2-linear operators on 2-normed sets*, Glas. Mat. Ser. III **39** (2004), 301–312.
11. R. Malcheski and Z. Cvetkovski, *Dual space of the space of bounded linear  $n$ -functionals*, Mat. Bilten **36** (2012), 47–53.
12. P.M. Miličić, *On the Gram-Schmidt projection in normed spaces*, Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat. **4** (1993), 85–92.
13. Y.E.P. Pangalela and H. Gunawan, *The  $n$ -dual space of the space of  $p$ -summable sequences*, Math. Bohem. **138** (2013), 439–448.
14. A.L. Soenjaya, *On  $n$ -bounded and  $n$ -continuous operator in  $n$ -normed space*, J. Indones. Math. Soc. **18** (2012), 45–56.
15. A. White, *2-Banach spaces*, Math. Nachr. **42** (1969), 43–60.
16. R.A. Wibawa-Kusumah and H. Gunawan, *Two equivalent  $n$ -norms on the space of  $p$ -summable sequences*, Period. Math. Hungar. **67** (2013), 63–69.

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF OTAGO, PO BOX 56,  
DUNEDIN 9054, NEW ZEALAND

*E-mail address:* yosafat.pangalela@maths.otago.ac.nz